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MULTIPLE CO-PRIME DIVISORS AND MULTIPLE DIVISORS GRAPHS

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ABSTRACT

In this paper we introduce two new concepts of graphs. Let R be a commutative ring with identity and Z(R)\{0} be the set of elements in R divided it into two sets $Z_1(R)$ and $Z_2(R)$, where $Z_1(R)$ be the set of co-prime divisor elements satisfies the Euler's function and $Z_2(R)$ be the set of non-co-prime divisor elements, a simple graph MC(R) is associated to Z(R) it is called multiple co-prime divisors graph. Moreover, Let R be a commutative ring with identity and Z*(R)=Z(R)\{0} be the set of all non-zero divisor elements in R, a simple graph M(R) is associated to R for distinct elements a and b in Z*(R) is an edge in M(R) if and only if aba=a(mod n) for all n in N. Also, diameter, girth, chromatic number and nullity of multiple co-prime divisor and multiple divisor graphs will be determined.

KEYWORDS: zero divisor graph, multiple co-prime divisor graphs and multiple divisor graphs.

1. INTRODUCTION

A graph is an ordered pair G=(V, E), where V is a nonempty set of vertices and E the set of edges of G. we shall consider G is undirected and finite. A path in G is a sequence $v_1,v_2,\dots,v_n$ of vertices where v_i is adjacent to v_{i+1}. A graph G is connected if every two vertices can be joined by a path. The distance between two vertices u and v is the minimum of the lengths of all u-v paths in G, and is denoted by $d_G(u, v)$. If no u-v path exists, we set $d_G(u, v)=\infty$. The girth of G is the length of a shortest cycle and is denoted by gir(G), and the circumference is the length of a longest cycle. Let G be a simple undirected graph with vertex set V = V(G) and edge set E = E(G). A graph G is said to be a **singular graph** provided that its adjacency matrix A(G) is a singular matrix. Let $\lambda_1, \lambda_2,\dots, \lambda_p$ be the eigenvalues of a graph G, which they form the spectrum of G. The algebraic multiplicity of the number zero in the spectrum of the graph G is called its **nullity (degree of singularity)** and denoted by $\eta(G)$ which was studied by Gutman in [2]. The concept of zero divisor graphs $\Gamma(R)$ was introduced by I. Beck in [4] but his motive was in coloring of graphs. In [3], Anderson and Livingston studied the diameter of a graph $\Gamma(R)$ of a commutative ring R with identity. For undefined terms in graph theory, we refer to [7].

Two vertices u and v in a graph G are said to be coneighbor vertices if and only if $N_G(u)=N_G(v)$.

Lemma 1.1:[6] **(Coneighbor Lemma)** For any pair of coneighbor vertices u and v in a graph G, $\eta(G)=\eta(G-u)+1=\eta(G-v)+1$.

Corollary 1.2: [1, p.234] **(End Vertex Corollary)** If G is a bipartite graph with an end vertex, and H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G)=\eta(H)$.

2. THE MULTIPLE CO-PRIME DIVISORS GRAPH OF THE RING R

In this section, we introduce a new concept of graph and study the properties.

Definition 2.1: Let R be a commutative ring with identity and Z(R)\{0} be the set of elements in R divided it in to two sets $Z_1(R)$ and $Z_2(R)$, where $Z_1(R)$ be the set of co-prime divisor elements satisfies the Euler's function and $Z_2(R)$ be the set of non-co-prime divisor elements, a simple graph MC(R) is associated to Z(R) where every element in $Z_1(R)$ is adjacent to every element in $Z_2(R)$ this graph is called **multiple co-prime divisors graph**.

The multiple co-prime divisors graph of the ring $R=Z_n$, n=pq where p<q. Notice

that, in this paper, we assume that p<q and p, q be prime numbers.

The multiple co-prime divisors set has two sets of vertices $Z_1(Z_{pq})$ and $Z_2(Z_{pq})$ the vertices in Z_1 (Z_{pq}) of order (p-1)(q-1) are non-adjacent and the vertices in $Z_2(Z_{pq})$ of order p+q-2 are non-adjacent, but each vertex in Z_1 (Z_{pq}) is adjacent to all vertices in $Z_2(Z_{pq})$.

Each vertex in $Z_1(Z_{pq})$ are of degree (p+q-2) and they are non-adjacent with all other vertices of the same set $Z_1(Z_{pq})$ and each vertex in $Z_2(Z_{pq})$ are of degree ((p-1)(q-1)) and they are non-adjacent with all other vertices of the same set $Z_2(Z_{pq})$.

Theorem 2.2: Let R= Z_{pq}, then $MC(Z_{pq})$ is isomorphic to complete bipartite graph $K_{(p-1)(q-1),\ q+p-2}$.

Proof: According to the two sets of the multiple co-prime divisors graph, Z_1 and Z_2, where $Z_1 . Z_2 \neq 0$, means that are non-adjacent together but Z_1 and Z_2 are adjacent with all other vertices u_i, i=1, 2, 3,…, (p-1)(q-1) (u_i represented the first type of vertices in Z_1) and v_j, j = 1, 2, 3, …, (p+q-2) (v_j represented the second type of vertices in Z_2) they are also non adjacent together, $u_i. v_j \neq 0$, implies that the multiple co-prime divisors graph $MC(Z_{pq})$ is complete bipartite graph $K_{(p-1)(q-1),\ q+p-2}$.

Proposition 2.3: The diameter of $MC(Z_{pq})$) is less than or equal to 2.

Proof: The multiple co-prime divisors set has two sets of vertices $Z_1(Z_{pq})$ and $Z_2(Z_{pq})$, each vertices in Z_1 (Z_{pq}) is adjacent to all vertices in $Z_2(Z_{pq})$, then the shortest path between u and v is of length one, so diam(u, v)=1, but the vertices in Z_1 (Z_{pq}) are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2, and the vertices in $Z_2(Z_{pq})$ are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2. Hence diam($MC(Z_{pq})$) $\leq$ 2.

Proposition 2.4: The girth of $MC(Z_{pq})$ is equal to 4.

Proof: The set of the multiple co-prime divisors graph $MC(Z_{pq})$ is $Z_1(Z_{pq})$ Let $u_1, u_2, u_3, …, u_i$, i=1, 2, 3,…, (p-1)(q-1) (u_i represented the first type of vertices in Z_1) and $v_1, v_2, v_3, …, v_j$, j = 1, 2, 3, …, (p+q-2) (v_j represented the second type of vertices in Z_2) respectively. Then all the cycles in the graph $MC(Z_{pq})$ are of length 4 , they are the smallest and they are of the form $u_1 \to v_1 \to u_2 \to v_2 \to u_1$. Then the girth is equal to 4.

Proposition 2.5: The chromatic number of $MC(Z_{pq})$ is equal to (p-1)(q-1).

Proof: Since no two adjacent vertices take the same color in coloring vertices in any graph, then each vertex in Z_1 have the different color and all other vertices u_i, i=1, 2, …, (p-1)(q-1) which represent the vertices in Z_1 has another color. Therefore the total number of colors used in the coloration of this graph is (p-1)(q-1), implies that the chromatic number is (p-1)(q-1).

Proposition 2.6: The multiple co-prime divisors graph of Z_{qp} is star graph $S_{1,\ p+q-2}$, by removing the (pq-p-q) from vertices of $Z_1(Z_{pq})$.

Proof: The vertices in the multiple co-prime divisors graph are two sets of vertices $Z_1(Z_{pq})$ and $Z_1(Z_{pq})$ the first set from 1 to (p-1)(q-1) and the second set from 1 to p+q-2, where each vertices in $Z_1(Z_{pq})$ are of degree (p+q-2) and they are non-adjacent with all other vertices of the same set $Z_1(Z_{pq})$, therefore, when we remove all the vertices in $Z_1(Z_{pq})$ except a vertex, say u_1, then we get the star graph of the form $S_{1,\ p+q-2}$.

Theorem 2.7: The nullity of $MC(Z_{pq})$ is equal to pq-4.

Proof: In the $MC(Z_{pq})$), we have two sets of coneighbor vertices, by using Lemma 1.1, we get: $\eta(MC(Z_{pq}))$= (p-1)(q-1)-1+q+p-2-1+$\eta(MC(K_2))$=pq-4+$\eta(MC(K_2))$ and $\eta(MC(K_2))$=0. Hence, $\eta(MC(Z_{pq}))$=pq-4.

The multiple co-prime divisors graph of the ring Z_n, $n=p^2q$ where p=2<q.

The multiple co-prime divisors set has two sets of vertices $Z_1(R)$ and $Z_2(R)$ the vertices in Z_1 (Z_{p^2q}) of order pq+5 are non-adjacent and the vertices in $Z_2(Z_{p^2q})$ of order 2p(q-1) are non-adjacent, but each vertex in Z_1 (Z_{p^2q}) is adjacent to all vertices in $Z_2(Z_{p^2q})$, each vertex in $Z_1(Z_{p^2q})$ are of degree 2p(q-1) and they are non-adjacent with all other vertices of the same set $Z_1(Z_{p^2q})$ and each vertex in $Z_2(Z_{p^2q})$ are of degree (pq+5) and they are non-adjacent with all other vertices of the same set $Z_2(Z_{p^2q})$.

Theorem 2.8: Let R= Z_{p^2q}, where p=2 <q, then $MC(Z_{p^2q})$ is isomorphic to complete bipartite graph $K_{pq+5,\ 2p(q-1)}$.

Proof: According to the two sets of the multiple co-prime divisors graph, Z_1 and Z_2, where $Z_1 . Z_2 \neq 0$, means that are non-adjacent together but Z_1 and Z_2 are adjacent with all other vertices u_i, i=1, 2, 3,…, pq+5 (u_i represented the first type of vertices in Z_1) and v_j, j = 1, 2, 3, …, 2p(q-1) (v_j represented the second type of vertices in Z_2) they are also non adjacent together, $u_i . v_j \neq 0$, implies that the multiple co-prime divisors graph $MC(Z_{p^2q})$ is complete bipartite graph $K_{pq+5,\ 2p(q-1)}$.

Proposition 2.9: The diameter of $MC(Z_{p^2q})$) is less than or equal to 2.

Proof: The multiple co-prime divisors set has two sets of vertices $Z_1(Z_{p^2q})$ and $Z_2(Z_{p^2q})$, each vertices in Z_1 (Z_{p^2q}) is adjacent to all vertices in $Z_2(Z_{p^2q})$, then the shortest path between u and v is of length one, so diam(u, v)=1, but the vertices in Z_1 (Z_{p^2q}) are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2, and the vertices in $Z_2(Z_{p^2q})$ are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2. Hence, diam($MC(Z_{p^2q})$) $\leq$ 2.

Proposition 2.10: The girth of $MC(Z_{p^2q})$ is equal to 4.

Proof: The set of the multiple co-prime divisors graph $MC(Z_{p^2q})$ is $Z_1(Z_{p^2q})$ Let $u_1, u_2, u_3, \ldots, u_i$, i=1, 2, 3,…, pq+5 (u_i represented the first type of vertices in Z_1) and $v_1, v_2, v_3, \ldots, v_j$, j = 1, 2, 3, …, 2p(q-1) (v_j represented the second type of vertices in Z_2) respectively. Then all the cycles in the graph $MC(Z_{p^2q})$ are of length 4, they are the smallest and they are of the form $u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow u_1$. Then the girth is equal to 4.

Proposition 2.11: The chromatic number of $MC(Z_{p^2q})$ is equal to 2.

Proof: Since no two adjacent vertices take the same color in coloring vertices in any graph, then each vertex in Z_1 have the same color and each vertex in Z_2 have the same color. Therefore the total number of colors used in the coloration of this graph is 2, implies that the chromatic number of $MC(Z_{p^2q})$ is equal to 2.

Proposition 2.12: The multiple co-prime divisors graph of Z_{p^2q} is star graph $S_{1,\ 2p(q-1)}$, by removing the pq+4 vertices from $Z_1(Z_{p^2q})$.

Proof: The vertices in the multiple co-prime divisors graph are two sets of vertices $Z_1(Z_{p^2q})$ and $Z_1(Z_{p^2q})$ the first set from 1 to pq+5and the second set from 1 to p+q-2, where each vertex in $Z_1(Z_{p^2q})$ are of degree 2p(q-1) and they are non-adjacent with all other vertices of the same set $Z_1(Z_{p^2q})$, therefore when we remove all the vertices in $Z_1(Z_{p^2q})$ except a vertex, say u_1, then we get the star graph $S_{1,\ 2p(q-1)}$.

Theorem 2.13: The nullity of $MC(Z_{p^2q})$ is equal to 6q-1, where p=2 <q.

Proof: In the $MC(Z_{p^2q})$), we have two sets of coneighbor vertices, by using Lemma 1.1, we get: $\eta(MC(Z_{p^2q}))$= pq+5+2p(q-1)-2+$\eta(K_2)$=3pq-2p+3+$\eta(K_2)$ and $\eta(K_2)$=0. Hence, $\eta(MC(Z_{2^2q}))$= 6q-1.

The multiple co-prime divisors graph of the ring Z_n, n=pqr where p=2<q=3<r.

The multiple co-prime divisors set has two sets of vertices $Z_1(R)$ and $Z_2(R)$ the vertices in Z_1 (Z_{pqr}) of order 2r-2 are non-adjacent and the vertices in $Z_2(Z_{pqr})$ of order r(q+1)+1 are non-adjacent, but each vertex in Z_1 (Z_{pqr}) is adjacent to all vertices in $Z_2(Z_{pqr})$, each vertex in $Z_1(Z_{pqr})$ are of degree (p+q-2) and they are non-adjacent with all other vertices of the same set $Z_1(Z_{pqr})$ and each vertex in $Z_2(Z_{pqr})$ are of degree (2r-2) and they are non-adjacent with all other vertices of the same set $Z_2(Z_{pqr})$.

Theorem 2.14: Let R= Z_{pqr}, where p <q, then $MC(Z_{pqr})$ is isomorphic to complete bipartite graph $K_{2r-2,\ r(q+1)+1}$.

Proof: According to the two sets of the multiple co-prime divisors graph, Z_1 and Z_2, where $Z_1 . Z_2 \neq 0$, means that are non-adjacent together but Z_1 and Z_2 are adjacent with all other vertices u_i, i=1, 2, 3,…, 2r-2 (u_i represented the first type of vertices in Z_1) and v_j, j = 1, 2, 3, …, r(q+1)+1 (v_j represented the second type of vertices in Z_2) they are also non adjacent together, $u_i . v_j \neq 0$, implies that the multiple co-prime divisors graph $MC(Z_{pqr})$ is complete bipartite graph $K_{2r-2,\ r(q+1)+1}$.

Proposition 2.15: The girth of $MC(Z_{pqr})$ is equal to 4.

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Proof: The set of the multiple co-prime divisors graph $MC(Z_{pqr})$ is $Z_1(Z_{pqr})$ Let $u_1, u_2, u_3, \ldots, u_i$, i=1, 2, 3,…, 2r-2 (u_i represented the first type of vertices in Z_1) and $v_1, v_2, v_3, \ldots, v_j$, j = 1, 2, 3, …, r(q+1)+1 (v_j represented the second type of vertices in Z_2) respectively. Then all the cycles in the graph $MC(Z_{pqr})$ are of length 4 , they are the smallest and they are of the form $u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow u_1$. Then the girth is equal to 4.

Proposition 2.16: The chromatic number of $MC(Z_{pqr})$ is equal to 2.

Proof: The multiple co-prime divisors graph $MC(Z_{pqr})$ is isomorphic to complete bipartite graph $K_{2r-2,\ r(q+1)+1}$. Then the chromatic number of $MC(Z_{pqr})$ is equal to 2.

Proposition 2.17: The diameter of $MC(Z_{pqr})$) is less than or equal to 2.

Proof: The multiple co-prime divisors set has two sets of vertices $Z_1(Z_{pqr})$ and $Z_2(Z_{pqr})$, each vertices in Z_1 (Z_{pqr}) is adjacent to all vertices in $Z_2(Z_{pqr})$, then the shortest path between u and v is of length one, so diam(u, v)=1, but the vertices in Z_1 (Z_{pqr}) are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2, and the vertices in $Z_2(Z_{pqr})$ are non-adjacent, then the shortest path between any two vertices u and v is of length two, so diam(u, v)=2. Hence, $diam(MC(Z_{pqr})) \leq 2$.

Proposition 2.18: The multiple co-prime divisors graph of Z_{pqr} is star graph $S_{1,\ r(q+1)+1}$, by removing the (2r-3) vertices from the vertices of $Z_1(Z_{pqr})$.

Proof: The vertices in the multiple co-prime divisors graph are two sets of vertices $Z_1(Z_{pqr})$ and $Z_1(Z_{pqr})$ the first set from 1 to 2r-2 and the second set from 1 to r(q+1)+1, where each vertices in $Z_1(Z_{pqr})$ are of degree r(q+1)+1 and they are non-adjacent with all other vertices of the same set $Z_1(Z_{pqr})$, therefore, when we remove all the vertices in $Z_1(Z_{pqr})$ except a vertex, say u_1, then we get the star graph $S_{1,\ r(q+1)+1}$.

Theorem 2.19: The nullity of $MC(Z_{pqr})$ is equal to r(q+3)-3, where $p=2 < q=3 < r$.

Proof: In the $MC(Z_{pqr})$, we have two sets of coneighbor vertices, by using Lemma 1.1, we get: $\eta(MC(Z_{pqr}))$= 2r-2+rq+r+1-2+$\eta(K_2)$= 3r+rq-3+$\eta(K_2)$ and $\eta(K_2)$=0. Hence, $\eta(MC(Z_{pqr}))$= r(q+3)-3.

3. THE MULTIPLE DIVISORS GRAPH OF THE RING R

In this section, we introduce the new concept of graph and study the properties.

Definition 3.1: Let R be a commutative ring with identity and $Z^*(R)=Z(R)/\{0\}$ be the set of all non-zero zero divisor elements in R, a simple graph M(R) is associated to R for distinct elements a and b in $Z^*(R)$ is an edge in M(R) if and only if aba=a(mod n) for all n in N, this graph is called **multiple divisor graphs**.

The multiple divisors graph of the ring Z_n, n=pq where p=2<q.

The multiple divisor graph of Z_{pq}, where $p=2 < q$ has the following properties:

1- The vertex set of multiple divisor graph of Z_{pq} is $V(M(Z_{pq}))=\{v_i$, i=1, 2, …,pq-1}.

2- The order and size of the multiple divisor graph of Z_{pq} are (pq-1) and ((2pq+3q-11)/2), respectively.

3- The vertex v_1 adjacent to all vertices in $M(Z_{pq})$ and $deg(v_1)$=pq-2.

4- The vertex v_q adjacent to q-3/2 odd vertices less than v_{q-1} and $deg(v_q)$=q-1/2.

5- The vertex v_{q-1} adjacent to even vertices greater than v_q and $deg(v_{q-1})$=q+1/2.

6- The vertex v_{q+1} adjacent to even vertices greater than v_{q+1} and $deg(v_{q+1})$=q+1/2.

7- The multiple divisors graph of Z_{pq} have q-2 coneighbor vertices of degree one (q-2 end vertices).

8- The even vertices less than v_{q-1} are non-adjacent and of degree one.

9- The odd vertices greater than v_{q+1} are non-adjacent and of degree one.

10- The vertices in (8) and (9) are non-adjacent vertices.

Theorem 3.2: Let R= Z_{pq}, where p <q, then $M(Z_{pq})$ is 3-partite.

Proof: The vertex set of multiple divisor graph of Z_{pq} can be partitioned into three sets $V_1=\{v_1\}$, V_2={ v_q, v_{q+1} and all other vertices of degree one} and V_3={ v_{q-1} and the vertices adjacent to v_q} . The vertices in each vertex set are non-adjacent. Clearly the multiple divisor graph of Z_{pq} is 3-partite graphs.

Proposition 3.3: The diameter of $M(Z_{pq})$) is less than or equal to 2.

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Proof: The vertex set of multiple divisor graph of Z_{pq} can be partitioned into three sets $V_1=\{v_1\}$, $V_2=\{v_q, v_{q+1}$ and all other vertices of degree one$\}$ and $V_3=\{v_{q-1}$ and the vertices adjacent to $v_q\}$, then the shortest path between vertex V_1 with V_2 and V_3 is of length one, but the distance between vertices in set V_2 with vertices in set V_3 is of length two, then diam(M $(Z_{pq})) \leq 2$.

Proposition 3.4: The girth of $M(Z_{pq})$ is equal to 3.

Proof: The set of the multiple divisors graph $M(Z_{pq})$ is $V(M(Z_{pq}))=\{v_i, i=1, 2, \ldots, pq-1\}$. Then all the cycles in the graph $M(Z_{pq})$ are of length 3 or 4, they are the smallest and they are of the form $v_1 \rightarrow v_{q-1} \rightarrow v_{q+1} \rightarrow v_1$. Then the girth is 3.

Proposition 3.5: The chromatic number of $M(Z_{pq})$ is equal to 3.

Proof: The vertex set of multiple divisor graph of Z_{pq} is 3-partite sets, then each set have different color. Hence the chromatic number of $M(Z_{pq})$ is equal to 3.

Proposition 3.6: The multiple divisors graph of Z_{qp} is star graph $S_{1,\ pq-5}$, by removing the vertices v_{q-1}, v_q and $v_{q+1}$of $M(Z_{pq})$.

Proof: The proof is similar to the proof of Proposition 2.18.

Theorem 3.7: The nullity of $M(Z_{pq})$ is equal to 2q-2, where $p=2 < q$.
Proof: In the $M(Z_{2.3})$, we have 2 coneighbor vertices adjacent with v_1, by using Lemma 1.1 and Corollary 1.2, then $\eta(M\ (Z_6))= 1+\eta(K_2))=1$ and in the $M(Z_{2.5})$, we have 3 coneighbor vertices adjacent with v_1, by using Coneighbor Lemma and end vertex corollary , then $\eta(M\ (Z_{10}))= 2+\eta(K_2)+\eta(K_3)=2$. If $q>5$, in the $M(Z_{pq})$, we have q-2 coneighbor vertices adjacent with v_1, ((q-3)/2) coneighbor vertices adjacent with $v_1$and v_q, and ((q-3)/2) coneighbor vertices adjacent with $v_1$and v_{q-1} and v_{q+1}, by using Lemmas 1.1 and Corollary 1.2, we get: $\eta(M\ (Z_{pq}))= q-3+ ((q-3)/2) + ((q-3)/2) -2+\eta(K_2)+\eta(K_3)= 2q-2+$ and $\eta(K_2)= \eta(K_3)=0$. Hence, $\eta(M(Z_{pq}))=2q-2$.

Question:

1- Construct $MC(Z_{p^2q})$ and find properties.
2- Construct $M(Z_{pq^2})$ and find properties.
3- Construct $M(Z_{pqr})$ and find properties.

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