NANO $S_B$-OPERATORS AND NANO $S_B$-CONTINUITY IN NANO TOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce the notion of nano $S_B$-operators in term of the class of nano $S_B$-open sets as such as nano $S_B$-interior and nano $S_B$-closur with their properties in nano topological spaces. After that, by using the class of nano $S_B$-open sets, we introduce the concept of nano $S_B$-continuity. Also, we study the relationship among some types of nano continuous functions in nano topological spaces.

KEYWORDS: Nano $S_B$-open sets, Nano $S_B$-operators, Nano continuity, Nano $S_B$-continuity.

1. INTRODUCTION

The concept of nano topological space introduced by Thivagar et al [2] with respect to a subset $X$ of $U$ as the universe. Then some types of nano open sets defined and introduced such as nano semi-open sets and nano $\alpha$-open sets by Thivagar et al [2]. Then nano $\beta$-open sets introduced by Revathy et al [3]. By using nano semi-open sets with nano $\beta$-open sets, nano $S_B$-open (briefly $nS_B$-open) sets introduced by Pirbal et al [4]. Later, nano continuity, nano $\alpha$-continuity, nano semi-continuity and nano $\beta$-continuity defined in [2] and [7]. In this work, we study nano $S_B$-continuity, but for this duty, we have to study nano $S_B$-operators; most importantly nano $S_B$-interior and nano $S_B$-closure.

2. PRELIMINARIES

Definition 2.1. [1] Let $U$ be a non-empty finite set of objects called the universe and $R$ be an equivalence relation on $U$ called as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U, R)$ is said to be the approximation space. Let $X \subseteq U$:

i. The lower approximation of $X$ with respect to $R$ is the set of all objects which can be for certain classified as $X$ with respect to $R$ and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in X} \{R(x); R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by $x$.

ii. The upper approximation of $X$ with respect to $R$ is the set of all objects which can be possibly classified as $X$ with respect to $R$ and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x); R(x) \cap X \neq \phi\}$.

iii. The boundary region of $X$ with respect to $R$ is the set of all objects which can be classified neither as $X$ nor as not-$X$ with respect to $R$ and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [2] Let $U$ be the universe and $R$ be an equivalence relation on $U$ and

$\tau_R(X) = \{\phi, U, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the followings.

i. $U$ and $\phi \in \tau_R(X)$

ii. The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.}

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iii. The intersection of elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
That is, $\tau_R(X)$ forms a topology on $U$ and called the nano topology on $U$ with respect to $X$. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the nano dual topology of $\tau_R(X)$.

Definition 2.3. Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The set $A$ is said to be:
i. Nano $\alpha$-open [2], if $A \subseteq \text{nint}(\text{ncl}(\{A\}))$.
ii. Nano semi-open [2], if $A \subseteq \text{ncl}(\text{nint}(A))$.
i. Nano $\beta$-open (nano semi pre-open) [3], if $A \subseteq \text{ncl}(\text{nint}(\text{ncl}(A)))$.

iv. Nano $S_S$-open [4], if $A$ is nano semi-open and $A = \cup \{F_n; F_n$ nano $\beta$-closed sets$\}$.
The family of all nano $\alpha$-open, nano semi-open, nano $\beta$-open and nano $S_S$-open sets denoted by $\text{nAO}(U, X), \text{nSO}(U, X), \text{nBO}(U, X)$ and $\text{nSO}(U, X)$ respectively.

Theorem 2.4. [7] Let $A$ be any subset of a nano topological space $(U, \tau_R(X))$, then:
i. $\text{nSint}(A) = \cup \{G; G$ is $\text{nS}$-open and $G \subseteq A\}$
ii. $\text{nScl}(A) = \cap \{F; F$ is $\text{nS}$-closed and $A \subseteq F\}$

Theorem 2.5. [6] Let $A$ be any subset of a nano topological space $(U, \tau_R(X))$, then:
i. $\text{nSint}(A) = \cup \{G; G$ is $\text{nS}$-open and $G \subseteq A\}$
ii. $\text{nScl}(A) = \cap \{F; F$ is $\text{nS}$-closed and $A \subseteq F\}$

Definition 2.6. Let $(U, \tau_R(X))$ and $(V, \tau'_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(V, \tau'_R(Y))$ is said to be:
i. Nano continuous [5], if $f^{-1}(A)$ is nano open set in $U$ for every nano open set $A$ in $V$.
ii. Nano semi-continuous [7], if $f^{-1}(A)$ is nano open set in $U$ for every nano $\alpha$-open set $A$ in $V$.
iii. Nano $\alpha$-continuous [7], if $f^{-1}(A)$ is nano open set in $U$ for every nano $\alpha$-open set $A$ in $V$.
iv. Nano $\beta$-continuous [6], if $f^{-1}(A)$ is nano open set in $U$ for every nano $\beta$-open set $A$ in $V$.

Theorem 2.7. [4] Let $(U, \tau_R(X))$ be a nano topological space, then the following statements are true:
i. Every $nS_S$-open set is $nS$-open.
i. Every $nS_S$-open set is $n\beta$-open.

Theorem 2.8. [4] If $U_R(X) = U$ and $L_R(X) = \phi$
in a nano topological space $(U, \tau_R(X))$, then

$v. \tau_R(X) = \tau_R(X)$.

Theorem 2.9. [4] If $U_R(X) = U$ and $L_R(X) \neq \phi$
in a nano topological space $(U, \tau_R(X))$, then

$\tau_R(X) = \tau_R(X)$.

Theorem 2.10. [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) = \{x\}$,

$\tau_R(X) = \tau_R(X)$.

Theorem 2.11. [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) \neq U$ and

$U_R(X)$ contains more than one element of $U$,
then the family of all $nS_S$-open sets in $U$ are $\phi$
and those sets $A$ for which $U_R(X) \subseteq A$.

Theorem 2.12. [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq U, L_R(X) = \phi$
and $U_R(X)$ contains more than one element of $U$,
then the family of all $nS_S$-open sets in $U$ are $\phi$
and those sets $A$ for which $U_R(X) \subseteq A$. 

x. Theorem 2.13. [4] Let \((U, \tau_\beta(X))\) be a nano
topological space. If \(U_\beta(X) \neq L_\beta(X)\) where
\(U_\beta(X) \neq U\) and \(L_\beta(X) \neq \emptyset\), then \(\emptyset\).
\(L_\beta(X), B_\beta(X), L_\beta(X) \cup B, B_\beta(X) \cup B\) and any
set containing \(U_\beta(X)\) where \(B \subseteq [U_\beta(X)]^c\) are
the only \(nS_\beta\)-open sets in \(U\).

topological space \((U, \tau_\beta(X))\) is \(nS_\beta\)-open if and
only if \(A\) is \(nS\)-open and is a union of \(n\beta\)-closed
sets.

xii. Proposition 2.15. [4] If a nano topological space
\((U, \tau_\beta(X))\) is locally indiscrete, then every \(nS\)-
open set is \(nS_\beta\)-open.

3. NANO \(S_\beta\)-OPERATORS

xiii. Definition 3.1. A subset \(N\) of a nano topological
space \((U, \tau_\beta(X))\) is said to be a nano \(S_\beta\-
neighborhood of a subset \(A\) of \(U\), if there exists
an \(nS_\beta\)-open set \(G\) such that \(A \subseteq G \subseteq N\), it is
denoted by \(nS_\beta\)-neighborhood.

xiv. Definition 3.2. A point \(x \in U\) is said to be a
nano \(S_\beta\)-interior point of a subset \(A\) of \(U\), if there
exists an \(nS_\beta\)-open set \(G\) containing \(x\) such that
\(x \in G \subseteq A\). The set of all \(nS_\beta\)-interior points of
\(A\) is said to be \(nS_\beta\)-interior of \(A\) and denoted by
\(nS_\beta\text{int}(A)\).

xv. Theorem 3.3. Let \(A\) be any subset of a nano
topological space \((U, \tau_\beta(X))\). If a point
\(x \in nS_\beta\text{int}(A)\), then there exists \(F \in n\beta\text{C}(U, X)\)
containing \(x\) such that \(F \subseteq A\).

Proof. Suppose that \(x \in nS_\beta\text{int }A\), then there
exists an \(nS_\beta\)-open set \(G\) containing \(x\) such that
\(G \subseteq A\). Since \(G \in nS_\beta\text{O}(U, X)\), then there exists
\(F \in n\beta\text{C}(U, X)\) containing \(x\) such that
\(F \subseteq G \subseteq A\). Hence \(x \in F \subseteq A\).

xvi. Theorem 3.4. Let \(A\) be any subset of a nano
topological space \((U, \tau_\beta(X))\), then:

i. \(nS_\beta\text{int}(A) \subseteq n\text{int}(A)\).

ii. \(nS_\beta\text{int}(A) \subseteq n\beta\text{int}(A)\).

Proof. Obvious.

xvii. The equality in Theorem 3.4. is not true in
general, as it shown in the following examples.

xviii. Example 3.5. Let \(U = \{a, b, c\}\) with
\(U/R = \{\{a\}, \{b, c\}\}\) and \(X = \{a\}\), then
\(\tau_\beta(X) = \{\emptyset, \{a\}\}\).
\(n\text{SO}(U, X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}\) and
\(n\beta\text{O}(U, X) = \{\emptyset, U\}\). Let \(A = \{a\}\), then
\(nS_\beta\text{int}(A) = \emptyset\) but \(n\text{int}(A) = \{a\}\). Therefore,
\(nS_\beta\text{int}(A) \neq n\text{int}(A) = n\beta\text{cl}(A)\).
Theorem 3.6. Let $A$ be any subset of a nano topological space $(U, \tau_{nR}(X))$, then:

i. $nS_{p}\text{int}(A) \subseteq A$.

ii. $nS_{p}\text{int}(A) = \bigcup \{G : G \text{ is } nS_{p}\text{-open and } G \subseteq A\}$.

iii. $A$ is a $nS_{p}\text{-open}$ if and only if $A = nS_{p}\text{int}(A)$.

iv. $nS_{p}\text{int}(nS_{p}\text{int}(A)) = nS_{p}\text{int}(A)$.

v. $nS_{p}\text{int}(\phi) = \phi$ and $nS_{p}\text{int}(U) = U$.

Proof. Straightforward.

The inclusion of part (i) of Theorem 3.6. cannot be replaced by equality in general, as it shown in the following example.

Example 3.7. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then

$\tau_{nR}(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}$ and

$nS_{p}\text{int}(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}\}$.

Let $A = \{d, b\}$, then $nS_{p}\text{int}(A) = \phi \neq A$.

Theorem 3.8. Let $A$ and $B$ be any two subset of a nano topological space $(U, \tau_{nR}(X))$, then:

i. If $A \subseteq B$, then $nS_{p}\text{int}(A) \subseteq nS_{p}\text{int}(B)$.

ii. $nS_{p}\text{int}(A) \cup nS_{p}\text{int}(B) \subseteq nS_{p}\text{int}(A \cup B)$.

iii. $nS_{p}\text{int}(A \cap B) \subseteq nS_{p}\text{int}(A) \cap nS_{p}\text{int}(B)$.

Proof. Straightforward.

The inclusion of parts (ii and iii) of Theorem 3.8. cannot be replaced by equality in general, as it is shown in the following example.

Example 3.9. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then

$\tau_{nR}(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}$ and

$nS_{p}\text{int}(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c, d\}\}$.

. For part (ii), take $A = \{b, d\}$ and $B = \{c, d\}$.

\[nS_{p}\text{int}(A) = \{c, d\}\]

then $nS_{p}\text{int}(A) \cup nS_{p}\text{int}(B) = \phi \cup \phi = \phi$ but

\[nS_{p}\text{int}(A \cup B) = \{b, c, d\}\]. Therefore,

\[nS_{p}\text{int}(A) \cup nS_{p}\text{int}(B) \neq nS_{p}\text{int}(A \cup B)\]. For part (iii), take $A = \{a, d\}$ and $B = \{b, c, d\}$, then

\[nS_{p}\text{int}(A \cap B) = nS_{p}\text{int}(\{d\}) = \phi\], but

\[nS_{p}\text{int}(A) \cap nS_{p}\text{int}(B) = \{a, d\} \cap \{b, c, d\} = \{d\}\].

Therefore,

\[nS_{p}\text{int}(A \cap B) \neq nS_{p}\text{int}(A) \cap nS_{p}\text{int}(B)\].

Corollary 3.10. Let $(U, \tau_{nR}(X))$ be a nano topological space when $U_{R}(X) = U$ and $L_{R}(X) = \phi$, then for any proper subset $A$ of $U$,

\[nS_{p}\text{int}(A) = \phi\].

Proof. Follows from Theorem 2.8. that $\phi$ and $U$ are the only $nS_{p}\text{-open}$ sets in $U$.

Corollary 3.11. Let $(U, \tau_{nR}(X))$ be a nano topological space when $U_{R}(X) = U$ and $L_{R}(X) = \phi$, then for any proper subset $A$ of $U$:

\[nS_{p}\text{int}(A) = \begin{cases} L_{R}(X), & \text{if } L_{R}(X) \subseteq A \\ B_{R}(X), & \text{if } B_{R}(X) \subseteq A \\ \phi, & \text{otherwise} \end{cases}\].

Proof. By Theorem 2.9., $\phi$, $U$, $L_{R}(X)$ and $B_{R}(X)$ are the only $nS_{p}\text{-open}$ sets in $U$, then it follows the result.
Corollary 3.12. Let \((U, \tau_R(X))\) be a nano topological space. If \(U_R(X) = L_R(X) = \{x\}\), \(x \in U\), then for any proper subset \(A\) of \(U\),\n\[nS_R \text{int}(A) = \phi.\]

Proof. Follows from Theorem 2.10.

Theorem 3.13. Let \((U, \tau_R(X))\) be a nano topological space. If \(U_R(X) = L_R(X) \neq U\) and \(U_R(X)\) contains more than one element of \(U\), then for any subset \(A\) of \(U\):
\[nS_R \text{int}(A) = \begin{cases} U_R(X), & \text{if } U_R(X) = A \\ A, & \text{if } U_R(X) \subset A \\ \phi, & \text{otherwise} \end{cases}\]

Proof. By Theorem 2.11, \(\phi\) and all sets \(A\) for which \(U_R(X) \subseteq A\) are the only \(nS_R\)-open sets in \(U\). So the result follows from the following cases:

i. \(U_R(X) = A\): \(nS_R \text{int}(A) = U_R(X)\).

ii. \(U_R(X) \subset A\): that is \(U_R(X)\) is a proper subset of \(A\), then \(A\) is \(nS_R\)-open set, thus \(nS_R \text{int}(A) = A\).

iii. \(U_R(X) \supset A\): then \(A\) is not \(nS_R\)-open, hence \(nS_R \text{int}(A) = \phi\).

Theorem 3.14. Let \((U, \tau_R(X))\) be a nano topological space. If \(U_R(X) \neq U\), \(L_R(X) = \phi\) and \(U_R(X)\) contains more than one element of \(U\), then for any subset \(A\) of \(U\):
\[nS_R \text{int}(A) = \begin{cases} U_R(X), & \text{if } U_R(X) = A \\ A, & \text{if } U_R(X) \subset A \\ \phi, & \text{otherwise} \end{cases}\]

Proof. The proof is similar to the proof of Theorem 3.13.

Definition 3.15. A point \(x \in U\) of a nano topological space \((U, \tau_R(X))\) is said to be \(S_R\)-cluster point of a subset \(A\) of \(U\), if \(A \cap G \neq \phi\) for every \(nS_R\)-open set \(G\) containing \(x\).

Definition 3.16. The set of all nano \(S_R\)-cluster points of a subset \(A\) of \(U\) is said to be \(nS_R\)-closure of \(A\) and it is denoted by \(nS_R \text{cl}(A)\).

Equivalently, The \(nS_R \text{cl}(A)\) is the intersection of all \(nS_R\)-closed sets containing \(A\).

Theorem 3.17. Let \(A\) be any subset of a nano topological space \((U, \tau_R(X))\). A point \(x \in nS_R \text{cl}(A)\) if and only if \(A \cap H \neq \phi\) for every \(nS_R\)-open set \(H\) containing \(x\).

Proof. Follows form Definition 3.15.

Corollary 3.18. For any subset \(A\) of a nano topological space \((U, \tau_R(X))\), the following statements are true.

i. \(nS_R \text{cl}(U - A) = U - nS_R \text{int}(A)\).

ii. \(nS_R \text{int}(U - A) = U - nS_R \text{cl}(A)\).

Proof. Obvious.

Theorem 3.19. For any subset \(A\) and \(B\) of a nano topological space \((U, \tau_R(X))\), the following statements are true:

i. If \(A \subseteq B\), then \(nS_R \text{cl}(A) \subseteq nS_R \text{cl}(B)\).

ii. \(nS_R \text{cl}(A) \cup nS_R \text{cl}(B) \subseteq nS_R \text{cl}(A \cup B)\).

iii. \(nS_R \text{cl}(A \cap B) \subseteq nS_R \text{cl}(A) \cap nS_R \text{cl}(B)\).

Proof. Straightforward.
The inclusion in (ii) and (iii) of the above theorem cannot be replaced by quality in general, as it is shown in the following example.

Example 3.20. Let \( U = \{a, b, c, d\} \) with
\[
U / R = \{\{a\}, \{b, c\}, \{d\}\} \quad \text{and} \quad X = \{a, b\},
\]
then
\[
\tau_R(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}.
\]
and
\[
nS_{p}O(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}
\]

For part (ii), take \( F = \{a, d\} \) and \( E = \{a, b\} \), then
\[
nS_{p}cl(F \cap E) = nS_{p}cl(\{a\}) = \{a\}, \quad \text{but} \quad nS_{p}cl(F) \cap nS_{p}cl(E) = \{a, d\} \cap \{a\} = \{a, d\}.
\]
Therefore,
\[
nS_{p}cl(F \cap E) \neq S_{p}cl(\{a\}) \cup nS_{p}cl(E).
\]
For part (iii), take \( F = \{b, c\} \) and \( E = \{a\} \), then
\[
nS_{p}cl(\{b, c\} \cup \{a\}) = U, \quad \text{but} \quad nS_{p}cl(\{b, c\}) \cup nS_{p}cl(\{a\}) = \{a, b, c\}.
\]
Therefore,
\[
nS_{p}cl(F) \cup nS_{p}cl(E) \neq nS_{p}cl(F \cup E).
\]

Theorem 3.21. Let \( A \) be any subset of a nano topological space \((U, \tau_{R}(X))\), then the following statements are true:

i. \( nS_{p}cl(\phi) = \phi \) and \( nS_{p}cl(U) = U \).

ii. \( A \subseteq nS_{p}cl(A) \).

iii. \( A \in nS_{p}C(U, X) \) if and only if \( A = nS_{p}cl(A) \).

iv. \( nS_{p}cl(nS_{p}cl(A)) = nS_{p}cl(A) \).

Proof. Straightforward.

Corollary 3.22. Let \((U, \tau_{R}(X))\) be a nano topological space when \( U_{R}(X) = U \) and \( L_{R}(X) = \phi \), then for any non-empty subset \( A \) of \( U \), \( nS_{p}cl(A) = U \).

Proof. Follows from Theorem 2.8.

Corollary 3.23. Let \((U, \tau_{R}(X))\) be a nano topological space when \( U_{R}(X) = U \) and \( L_{R}(X) \neq \phi \), then for any non-empty subset \( A \) of \( U \):
\[
nS_{p}cl(A) = \begin{cases} [L_{R}(X)]^c, & \text{if } A \subseteq L_{R}(X) \\ [B_{R}(X)]^c, & \text{if } A \subseteq B_{R}(X) \\ U, & \text{otherwise} \end{cases}
\]

Proof. By Theorem 2.9, we have \( \phi, U, L_{R}(X) \) and \( B_{R}(X) \) are the only \( nS_{p} \)-open sets in \( U \), then \( \phi, U, [L_{R}(X)]^c \) and \( [B_{R}(X)]^c \) are the only \( nS_{p} \)-closed sets in \( U \), so we get the result.

Corollary 3.24. Let \((U, \tau_{R}(X))\) be a nano topological space. If \( U_{R}(X) = L_{R}(X) = \{x\} \), \( x \in U \), then for any non-empty subset \( A \) of \( U \), \( nS_{p}cl(A) = U \).

Proof. Follows from Theorem 2.10.

Corollary 3.25. Let \((U, \tau_{R}(X))\) be a nano topological space. If \( U_{R}(X) = L_{R}(X) \neq U \) and \( U_{R}(X) \) contains more than one element of \( U \), then for any non-empty subset \( A \) of \( U \):
Proof. By Theorem 2.11, the only $nS_R$-closed sets are $\phi$ and those subsets $A$ for which $A \subseteq [U_R(X)]^c$. If $A = [U_R(X)]^c$, then $nS_Rcl([U_R(X)]^c) = [U_R(X)]^c$. If $A \subset [U_R(X)]^c$, then $A$ is $nS_R$-closed in $U$, hence $nS_Rcl(A) = A$. If $[U_R(X)]^c \subseteq A$, since $[U_R(X)]^c$ is the largest $nS_R$-closed set in $U$, hence $nS_Rcl(A) = U$.

Corollary 3.26. Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq U, L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $U$,

then for any non-empty subset $A$ of $U$:

$$nS_Rcl(A) = \begin{cases} [U_R(X)]^c, & \text{if } A = [U_R(X)]^c \\ A, & \text{if } A \subset [U_R(X)]^c \\ U, & \text{otherwise} \end{cases}$$

Proof. The proof is similar to Corollary 3.25.

4. NANO $S_R$-CONTINUOUS FUNCTIONS

Definition 4.1. A function

$$f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$$

is said to be $nS_R$-continuous at a point $x \in U$, if for each nano open set $B$ in $V$ containing $f(x)$, there exists an $nS_R$-open set $A$ in $U$ containing $x$ such that $f(A) \subseteq B$. If $f$ is $nS_R$-continuous at every point $x$ of $U$, then it is called $nS_R$-continuous.

Definition 4.2. A function

$$f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$$

is said to be $nS_R$- irresolute if the inverse image of every $nS_R$-open set is $nS_R$-open.

Theorem 4.3. Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be a function, then the following statements are equivalent:

1. $f$ is $nS_R$-continuous function.
2. The inverse image of every nano open set $B$ in $V$ is $nS_R$-open set in $U$.
3. The inverse image of every nano closed set $F$ in $V$ is $nS_R$-closed set in $U$.
4. For each $A \subseteq U, f(nS_Rcl(A)) \subseteq ncl(f(A))$.
5. For each $A \subseteq U, nint(f(A)) \subseteq f(nS_Rint(A))$.
6. For each $B \subseteq V, nS_Rcl(f^{-1}(B)) \subseteq f^{-1}(ncl(B))$.
7. For each $B \subseteq V, f^{-1}(nint(B)) \subseteq nS_Rint(f^{-1}(B))$.

Proof. Straightforward.

Example 4.4. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, c\}$. Then $\tau_R(X) = \{\phi, U, \{c\}, \{a, b, c\}, \{a, b\}\}$ and $nS_Rcl(X) = \{\phi, U, \{c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c\}\}$.

Let $V = \{x, y, w, z\}$ with $V/R' = \{\{x\}, \{z\}, \{y, w\}\}$ and $V = \{x, z\}$. Then $\tau_{R'}(Y) = \{\phi, V, \{x\}, \{y, w\}, \{x, y, w\}\}$. Define $f: U \rightarrow V$ as $f(a) = y, f(b) = w, f(c) = x$ and $f(d) = z$, then $f$ is $nS_R$-continuous function.

For part (4), take $A = \{a\}$, then $\{y, w\} = f(nS_Rcl(A)) \supseteq ncl(f(A)) = \{y, w, z\}$. 


ii. For part (5), take $A = \{c, d\}$, then
\[
\{x\} = \text{nt}(f(A)) \cap f(nS_{\beta}\text{int}(A)) = \{x, z\}.
\]

iii. For part (6), take $F = \{x\}$, then,
\[
\{c\} = nS_{\beta}\text{cl}(f^{-1}(B)) = f^{-1}(\text{nc}(B)) = \{c, d\}.
\]

iv. For part (7), take $F = \{y, w, z\}$, then
\[
\{a, b\} = f^{-1}(\text{int}(B)) \cap nS_{\beta}\text{int}(f^{-1}(B)) = \{a, b, d\}.
\]

**Remark 4.5.** Nano continuous function and $nS_{\beta}$-continuous function are independent in general, as it is shown by the following example.

**Example 4.6.** Let
\[
U = \{a, b, c\} \text{ with } U/R = \{\{a\}, \{b\}, \{c\}\} \text{ and }
\]
\[
X = \{a\}, \text{ then } \tau_{\alpha}(X) = \{\phi, U, \{a\}\}, \text{ then }
\]
\[
nS_{\beta}O(X) = \{\phi, U\}. \text{ Define the identity function }
\]
\[
f: (U, \tau_{\beta}(X)) \to (U, \tau_{\beta}(X)), \text{ consider }
\]
\[
f^{-1}(\{a\}) = \{a\} \in \tau_{\beta}(X) \text{ but } \{a\} \notin nS_{\beta}O(X).
\]

Therefore, $f$ is nano continuous but not $nS_{\beta}$-continuous function. Also, let $V = \{a, b, c, d\}$ with $V/R = \{\{a, b\}, \{c, d\}\}$ and $X = \{a, b\}$, then
\[
\tau_{\alpha}(X) = \{\phi, V, \{a, b\}\} \text{ and }
\]
\[
nS_{\beta}O(V, X) = \{\phi, V, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}.
\]

Define the function $f: (V, \tau_{\beta}(X)) \to (V, \tau_{\beta}(X))$
\[
\text{by } f(a) = a, f(b) = a, f(c) = b \text{ and } f(d) = d,
\]

then $f^{-1}(\{a, b\}) = \{a, b, c\} \notin \tau_{\beta}(X)$. Therefore, $f$ is not nano continuous but $nS_{\beta}$-continuous function.

**Proposition 4.7.** Let
\[
f: (U, \tau_{\beta}(X)) \to (V, \tau_{\beta}(Y)) \text{ be a function, then }
\]
the following statements are true:

lix. Every $nS_{\beta}$-continuous function is $nS$-continuous.

lx. Every $nS_{\beta}$-continuous function is $nS$-continuous.

**Proof.** Obvious.

lxi. Example 4.6 shows that the converse of above proposition is not true in general.

lxii. **Proposition 4.8.** If $f: (U, \tau_{R}(X)) \to (V, \tau_{R}(Y))$ is $nS$-continuous function and $U$ is locally indiscrete, then $f$ is $nS_{\beta}$-continuous.

**Proof.** Suppose that $f$ is $nS$-continuous function and $B$ be any nano open set in $V$. Then $f^{-1}(B)$ is $nS$-open subset of $U$, since $U$ is locally indiscrete, then by Proposition , $f^{-1}(B) \in nS_{\beta}O(U, X)$. Hence $f$ is $nS_{\beta}$-continuous function.

lxiii. **Theorem 4.9.** A function
\[
f: (U, \tau_{R}(X)) \to (V, \tau_{R}(Y)) \text{ is } nS_{\beta}-\text{continuous if and only if } f \text{ is } nS-\text{continuous and for each } x \in U \text{ and each nano open set } B \text{ of } V \text{ containing } f(x), \text{ there exists an } n\beta-\text{closed set } F \text{ in } U
\]
containing $x$ such that $f(F) \subseteq B$.

**Proof.** Let $B$ be any nano open set in $V$. Since $f$ is $nS$-continuous, then $f^{-1}(B)$ is $nS$-open set in $U$. Let $x \in f^{-1}(B)$, so $f(x) \in B$. By assumption, there exists a $n\beta$-closed set $F$ of $U$ such that $f(F) \subseteq B$. Which it implies that $x \in F \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B)$ is a $nS_{\beta}$-open set in $U$. Hence $f$ is $nS_{\beta}$-continuous function.
Conversely, follows from Proposition 4.7.

lxiv. **Theorem 4.10.** Let $f: (U, \tau_{R}(X)) \to (V, \tau_{R}(Y))$ be an $nS_{\beta}$-continuous function and $U$ is...
extremely disconnected, then \( f \) is \( n\alpha \)-continuous.

**Proof.** Let \( f \) be \( nS_{\beta} \)-continuous function and \( A \) be any nano open subset of \( V \). Since \( f \) is \( nS_{\beta} \)-continuous, then \( f^{-1}(A) \in nS_{\beta}O(U,X) \) by Proposition, \( f^{-1}(A) \in nSO(U,X) \) and then \( f^{-1}(A) = \bigcup_{\lambda \in A} F_\lambda \), where \( F_\lambda \in n\beta G(U,X) \) for each \( \lambda \in A \), but since \( U \) is extremely disconnected, so
\[ f^{-1}(A) \subseteq ncl \left( nint \left( f^{-1}(A) \right) \right) = nint \left( ncl \left( f^{-1}(A) \right) \right) \]
Therefore, \( f^{-1}(A) \in n\alpha O(U,X) \). Thus, \( f \) is \( n\alpha \)-continuous.

**Theorem 4.11.** If the function \( f: (U, \tau_{R}(X)) \rightarrow (V, \tau_{R'}(Y)) \) is \( n\alpha \)-continuous function and \( U \)

is locally indiscrete, then \( f \) is \( nS_{\beta} \)-continuous.

**Proof.** Let \( f \) is \( n\alpha \)-continuous function, then the \( f^{-1}(G) \) is \( n\alpha \)-open in \( U \) for every nano open set \( G \) in \( V \). Since \( f^{-1}(G) \) is also \( n\beta \)-open and \( U \) is locally indiscrete, by Proposition 4.8, \( f \) is a \( nS_{\beta} \)-continuous function.

**Theorem 4.12.** The identity function \( f: (U, \tau_{R}(X)) \rightarrow (U, \tau_{R}(X)) \) is \( nS_{\beta} \)-continuous function if \( (U, \tau_{R}(X)) \) is extremely disconnected.

**Proof.** Obvious.

**Theorem 4.13.** A function \( f: (U, \tau_{R}(X)) \rightarrow (V, \tau_{R'}(Y)) \) is \( nS_{\beta} \)-continuous function if \( U_{R'}(Y) = V \) and \( L_{R'}(Y) = \phi \).

**Proof.** \( \tau_{R'}(Y) = nS_{\beta}(V,Y) = \{\phi, U\} \). Then \( f^{-1}(\phi) = \phi \) and \( f^{-1}(U) = U \). Hence \( f \) is \( nS_{\beta} \)-continuous function.

**Theorem 4.14.** The identity function \( f: (U, \tau_{R}(X)) \rightarrow (U, \tau_{R}(X)) \) is \( nS_{\beta} \)-continuous function if:

i. \( U_{R}(X) = U \) and \( L_{R}(X) = \phi \).

ii. \( U_{R}(X) = U \) and \( L_{R}(X) \neq \phi \).

iii. \( U_{R}(X) = L_{R}(X) \neq U \) and \( U_{R}(X) \) contains more than one element of \( U \).

iv. \( U_{R}(X) \neq U \), \( L_{R}(X) = \phi \) and \( U_{R}(X) \) contains more than one element of \( U \).

v. \( U_{R}(X) \neq L_{R}(X) \) where \( U_{R}(X) \neq U \) and \( L_{R}(X) = \phi \).

**Proof.**
function in general, as it is shown by the following example.

**Example 4.16.** Let \( U = \{a, b, c, d, e\} \) with \( U/R = \{(a, b), (c, d), (e)\} \) and \( X = \{a, c, d\} \).

Also, let \( U/R' = \{(a), (b, c), (d), (e)\} \) and

\[ nS_{nR}O(U, X) = \{ \phi, U, \{a, b, c, d\}, \{a, b\}, \{c, d, e\}, \{a, b, e\}\} \text{ and} \]

\[ nS_{nR}O(U, Y) = \{ \phi, U, \{a, b, c, d\}, \{a, d\}, \{b, e\}\} \]

**Definition.** Define the function \( f: (U, \tau_{nR}(X)) \rightarrow (U, \tau_{nR'}(Y)) \) by

\[ f(a) = b, f(b) = c, f(c) = a, f(d) = d, \text{ and} \]

\[ f(e) = e. \]

Define \( g: (U, \tau_{nR'}(Y)) \rightarrow (U, \tau_{nR}(X)) \) by \( g(a) = d, g(b) = a, g(c) = b, g(d) = c, \text{ and} \)

\[ g(e) = a. \]

It is clear \( f \) and \( g \) are \( nS_{nR} \)-continuous functions. Then

\[ h = fog: (U, \tau_{nR}(X)) \rightarrow (U, \tau_{nR}(X)) \]

\[ h(a) = d, h(b) = b, h(c) = c, h(d) = a \text{ and} \]

\[ h(e) = b. \]

Since \( \{c, d\} \in \tau_{nR}(X) \) but

\[ h^{-1}(\{c, d\}) \notin nS_{nR}O(U, X). \]

Hence, the composition of \( nS_{nR} \)-continuous functions need to be \( nS_{nR} \)-continuous.

**Theorem 4.17.** Let \( f: (U, \tau_{nR}(X)) \rightarrow (V, \tau_{nR'}(Y)) \) be \( nS_{nR} \)-continuous and

\( g: (V, \tau_{nR'}(Y)) \rightarrow (W, \tau_{nR''}(Z)) \) be a nano continuous functions. Then the composition functions \( fog: (U, \tau_{nR}(X)) \rightarrow (W, \tau_{nR''}(Z)) \) is \( nS_{nR} \)-continuous.

**Proof.** Let \( H \) be any nano open subset of \( (W, \tau_{nR''}(Z)) \). Since \( g \) is nano continuous, then \( g^{-1}(H) \) is a nano open subset of \( (V, \tau_{nR'}(Y)) \). Since \( f \) is \( nS_{nR} \)-continuous, then by Theorem 4.3., \( (fog)^{-1}(H) = f^{-1}(g^{-1}(H)) \) is \( nS_{nR} \)-open subset in \( (U, \tau_{nR}(X)) \). Therefore \( fog \) is \( nS_{nR} \)-continuous function.

5. CONCLUSION

In this paper, we have introduced the nano \( nS_{nR} \)-operators by using nano \( S_{nR} \)-open sets such as nano \( S_{nR} \)-interior and nano \( S_{nR} \)-closure with their properties and they are clarified by \( U_{nR}(X), \ L_{nR}(X) \) and \( B_{nR}(X) \) approximations. Also, the concept of nano \( S_{nR} \)-continuity have introduced and the relationship among some types of nano continuous function in nano topological spaces are considered such as we can see that nano continuous function and \( nS_{nR} \)-continuous function are independent in general. Also, we have shown that every \( nS_{nR} \)-continuous function is \( nS \)-continuous but the converse may not be true. Furthermore, it is shown that the composition of two \( nS_{nR} \)-continuous functions need to be \( nS_{nR} \)-continuous function in general.
6. REFERENCES