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NANo S_β-OPERATORS AND NANO S_β-CONTINUITY IN NANO TOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce the notion of nano nS_β-operators in term of the class of nano S_β-open sets such as nano S_β-interior and nano S_β-closure with their properties in nano topological spaces. After that, by using the class of nano S_β-open sets, we introduce the concept of nano S_β-continuity. Also, we study the relationship among some types of nano continuous functions in nano topological spaces.

KEYWORDS: Nano S_β-open sets, Nano S_β-operators, Nano continuity, Nano S_β-continuity.

1. INTRODUCTION

The concept of nano topological space introduced by Thivagar et al [2] with respect to a subset X of U as the universe. Then some types of nano open sets defined and introduced such as nano semi-open sets and nano α-open sets by Thivagar et al [2]. Then nano β-open sets introduced by Revathy et al [3]. By using nano semi-open sets with nano β-open sets, nano S_β-open (briefly nS_β-open) sets introduced by Pirbal et al [4]. Later, nano continuity, nano α-continuity, nano semi-continuity and nano β-continuity defined in [2] and [7]. In this work, we study nano S_β-continuity, but for this duty, we have to study nano S_β-operators; most importantly nano S_β-interior and nano S_β-closure.

2. PRELIMINARIES

i. **Definition 2.1.** [1] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U called as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$:

i. The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U} \{R(x); R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.

ii. The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U} \{R(x); R(x) \cap X \neq \phi\}$.

iii. The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

ii. **Definition 2.2.** [2] Let U be the universe and R be an equivalence relation on U and $\tau_R(X) = \{\phi, U, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the followings axioms:

i. U and $\phi \in \tau_R(X)$

ii. The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

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iii. The intersection of elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
That is, $\tau_R(X)$ forms a topology on U and called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the nano dual topology of $\tau_R(X)$.

iii.**Definition 2.3.** Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The set A is said to be:

i. Nano α-open [2], if $A \subseteq nint\left(ncl\left(nint((A))\right)\right)$.
ii. Nano semi-open [2], if $A \subseteq ncl(nint(A))$.
iii. Nano β-open (nano semi pre-open) [3], if $A \subseteq ncl\left(nint(ncl(A))\right)$.
iv. Nano S_β-open [4], if A is nano semi-open and $A = \cup \{F_\alpha;\ F_\alpha \text{ nano } \beta\text{-closed sets}\}$.

The family of all nano α-open, nano semi-open, nano β-open and nano S_β-open sets denoted by $n\alpha O(U, X)$, $nSO(U, X)$, $n\beta O(U, X)$ and $nS_\beta O(U, X)$ respectively.

Theorem 2.4. [7] Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:

i. $nSint(A) = \cup \{G: G \text{ is } nS\text{-open and } G \subseteq A\}$
ii. $nScl(A) = \cap \{F: F \text{ is } nS\text{-closed and } A \subseteq F\}$

Theorem 2.5. [6] Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:

i. $n\beta int(A) = \cup \{G: G \text{ is } n\beta\text{-open and } G \subseteq A\}$
ii. $n\beta cl(A) = \cap \{F: F \text{ is } n\beta\text{-closed and } A \subseteq F\}$

Definition 2.6. Let $(U, \tau_R(X))$ and $(V, \tau'_R(X))$ be two nano topological spaces. A function $f: (U, \tau_R\ (V, \tau_{R'}(Y))$ is said to be:

i. Nano continuous [5], if $f^{-1}(A)$ is nano open set in U for every nano open set A in V.
ii. Nano semi-continuous [7], if $f^{-1}(A)$ is nano open set in U for every nS-open set A in V.
iii. Nano α-continuous [7], if $f^{-1}(A)$ is nano open set in U for every $n\alpha$-open set A in V.
iv. Nano β-continuous [6], if $f^{-1}(A)$ is nano open set in U for every $n\beta$-open set A in V.

iv.**Theorem 2.7.** [4] Let $(U, \tau_R(X))$ is a nano topological space, then the following statements are true:

i. Every nS_β-open set is nS-open.
ii. Every nS_β-open set is $n\beta$-open.

v.**Theorem 2.8.** [4] If $U_R(X) = U$ and $L_R(X) = \phi$ in a nano topological space $(U, \tau_R(X))$, then $nS_\beta O(U, X) = \{U, \phi\}$.

vi.**Theorem 2.9.** [4] If $U_R(X) = U$ and $L_R(X) \neq \phi$ in a nano topological space $(U, \tau_R(X))$, then $\tau_R(X) = \tau_R^{S_\beta}(X)$.

vii.**Theorem 2.10.** [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) = \{x\}$, $x \in U$, then $nS_\beta O(U, X) = \{\phi, U\}$.

viii.**Theorem 2.11.** [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U, then the family of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.

ix.**Theorem 2.12.** [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U, then the family of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.

x. **Theorem 2.13.** [4] Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R(X) \neq \phi$, then ϕ, $L_R(X), B_R(X), L_R(X) \cup B, B_R(X) \cup B$ and any set containing $U_R(X)$ where $B \subseteq [U_R(X)]^c$ are the only nS_β-open sets in U.

xi. **Proposition 2.14.** [4] A subset A of a nano topological space $(U, \tau_R(X))$ is nS_β-open if and only if A is nS-open and is a union of $n\beta$-closed sets.

xii. **Proposition 2.15.** [4] If a nano topological space $(U, \tau_R(X))$ is locally indiscrete, then every nS-open set is nS_β-open.

3. NANO S_β-OPERATORS

xiii. **Definition 3.1.** A subset N of a nano topological space $(U, \tau_R(X))$ is said to be a nano S_β-neighborhood of a subset A of U, if there exists an nS_β-open set G such that $A \subseteq G \subseteq N$, it is denoted by nS_β-neighborhood.

xiv. **Definition 3.2.** A point $x \in U$ is said to be a nano S_β-interior point of a subset A of U, if there exists an nS_β-open set G containing x such that $x \in G \subseteq A$. The set of all nS_β-interior points of A is said to be nS_β-interior of A and denoted by $nS_\beta int(A)$.

xv. **Theorem 3.3.** Let A be any subset of a nano topological space $(U, \tau_R(X))$. If a point $x \in nS_\beta int(A)$, then there exists $F \in n\beta C(U, X)$ containing x such that $F \subseteq A$.

Proof. Suppose that $x \in nS_\beta int\ A$, then there exists an nS_β-open set G containing x such that $G \subseteq A$. Since $G \in nS_\beta O(U, X)$, then there exists $F \in n\beta C(U, X)$ containing x such that $F \subseteq G \subseteq A$. Hence $x \in F \subseteq A$.

xvi. **Theorem 3.4.** Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:

i. $nS_\beta int(A) \subseteq nSint(A)$.
ii. $nS_\beta int(A) \subseteq n\beta int(A)$.

Proof. Obvious.

xvii. The equality in Theorem 3.4. is not true in general, as it shown in the following examples.

xviii. **Example 3.5.** Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$, then $\tau_R(X) = \{\phi, U, \{a\}\}$, $nSO(U, X) = \{\phi, U, \{a\}, \{a, b\}, \{a, c\}\}$ and $nS_\beta O(U, X) = \{\phi, U\}$. Let $A = \{a\}$, then $nS_\beta int(A) = \phi$ but $nSint(A) = \{a\}$. Therefore, $nS_\beta int(A) \neq nSint(A) = n\beta cl(A)$.

xix. **Theorem 3.6.** Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:

i. $nS_\beta int(A) \subseteq A$.
ii. $nS_\beta int(A) = \cup \{G: G \text{ is } nS_\beta\text{-open and } G \subseteq A\}$
iii. A is a nS_β-open if and only if $A = nS_\beta int(A)$.
iv. $nS_\beta int\left(nS_\beta int(A)\right) = nS_\beta int(A)$.
v. $nS_\beta int(\phi) = \phi$ and $nS_\beta int(U) = U$.

Proof. Straightforward.

xx. The inclusion of part (i) of Theorem 3.6. cannot be replaced by equality in general, as it shown in the following example.

xxi. **Example 3.7.** Let $U = \{a, b, c, d\}$ with

$U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then

$\tau_R(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}$ and

$nS_\beta O(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}\}$

. Let $A = \{d, b\}$, then $nS_\beta int(A) = \phi \not\supseteq A$.

xxii. **Theorem 3.8.** Let A and B be any two subset of a nano topological space $(U, \tau_R(X))$, then:

i. If $A \subseteq B$, then $nS_\beta int(A) \subseteq nS_\beta int(B)$.
ii. $nS_\beta int(A) \cup nS_\beta int(B) \subseteq nS_\beta int(A \cup B)$.
iii. $nS_\beta int(A \cap B) \subseteq nS_\beta int(A) \cap nS_\beta int(B)$.

Proof. Straightforward.

xxiii. The inclusion of parts (ii and iii) of Theorem 3.8. cannot be replaced by equality in general, as it is shown in the following example.

xxiv. **Example 3.9.** Let $U = \{a, b, c, d\}$ with

$U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then

$\tau_R(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}$ and

$nS_\beta O(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}\}$

. For part (ii), take $A = \{b, d\}$ and $B = \{c, d\}$, then $nS_\beta int(A) \cup nS_\beta int(B) = \phi \cup \phi = \phi$ but $nS_\beta int\,(A \cup B) = \{b, c, d\}$. Therefore, $nS_\beta int(A) \cup nS_\beta int(B) \neq nS_\beta int(A \cup B)$. For part (iii), take $A = \{a, d\}$ and $B = \{b, c, d\}$, then $nS_\beta int(A \cap B) = nS_\beta int(\{d\}) = \phi$, but $nS_\beta int(A) \cap nS_\beta int(B) = \{a, d\} \cap \{b, c, d\} = \{d\}$

.

xxv. Therefore, $nS_\beta int(A \cap B) \neq nS_\beta int(A) \cap nS_\beta int(B)$.

xxvi. **Corollary 3.10.** Let $(U, \tau_R(X))$ be a nano topological space when $U_R(X) = U$ and $L_R(X) = \phi$, then for any proper subset A of U, $nS_\beta int(A) = \phi$.

Proof. Follows from Theorem 2.8. that ϕ and U are the only nS_β-open sets in U.

xxvii. **Corollary 3.11.** Let $(U, \tau_R(X))$ be a nano topological space when $U_R(X) = U$ and $L_R(X) \neq \phi$, then for any proper subset A of U:

$$nS_\beta int(A) = \begin{cases} L_R(X), & if\ L_R(X) \subseteq A \\ B_R(X), & if\ B_R(X) \subseteq A \\ \phi, & otherwise \end{cases}$$

xxviii. **Proof.** By Theorem 2.9., $\phi, U, L_R(X)$ and $B_R(X)$ are the only nS_β-open sets in U, then it follows the result.

xxix. **Corollary 3.12.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) = \{x\}$, $x \in U$, then for any proper subset A of U, $nS_\beta int(A) = \phi$.

xxx. **Proof.** Follows from Theorem 2.10.

xxxi. **Theorem 3.13.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U, then for any subset A of U:

$$nS_\beta int(A) = \begin{cases} U_R(X), & if\ U_R(X) = A \\ A, & if\ U_R(X) \subset A \\ \phi, & otherwise \end{cases}$$

xxxii. **Proof.** By Theorem 2.11. ϕ and all sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in U. So the result follows from the following cases:

i. If $U_R(X) = A$, $nS_\beta int(A) = U_R(X)$.

ii. If $U_R(X) \subset A$, that is $U_R(X)$ is a proper subset of A, then A is nS_β-open set, thus $nS_\beta \mathrm{int}(A) = A$.

iii. If $U_R(X) \supset A$, then A is not nS_β-open set, hence $nS_\beta int(A) = \phi$.

xxxiii. **Theorem 3.14.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U, then for any subset A of U:

$$nS_\beta int(A) = \begin{cases} U_R(X), & if\ U_R(X) = A \\ A, & if\ U_R(X) \subset A \\ \phi, & otherwise \end{cases}$$

xxxiv. **Proof.** The proof is similar to the proof of Theorem 3.13.

xxxv. **Definition 3.15.** A point $x \in U$ of a nano topological space $(U, \tau_R(X))$ is said to be nano S_β-cluster point of a subset A of U, if $A \cap G \neq \phi$ for every nS_β-open set G containing x.

xxxvi. **Definition 3.16.** The set of all nano S_β-cluster points of a subset A of U is said to be nS_β-closure of A and it is denoted by $nS_\beta cl(A)$. Equivalently, The $nS_\beta cl(A)$ is the intersection of all nS_β-closed sets containing A.

xxxvii. **Theorem 3.17.** Let A be any subset of a nano topological space $(U, \tau_R(X))$. A point $x \in nS_\beta cl(A)$ if and only if $A \cap H \neq \phi$ for every nS_β-open set H containing x.

Proof. Follows form Definition 3.15.

xxxviii. **Corollary 3.18.** For any subset A of a nano topological space $(U, \tau_R(X))$, the following statements are true.

i. $nS_\beta cl\ (U - A) = U - nS_\beta int(A)$.

ii. $nS_\beta int\ (U - A) = U - nS_\beta cl(A)$.

Proof. Obvious.

xxxix. **Theorem 3.19.** For any subset A and B of a nano topological space $(U, \tau_R(X))$, the following statements are true:

i. If $A \subseteq B$, then $nS_\beta cl(A) \subseteq nS_\beta cl(B)$.

ii. $nS_\beta cl(A) \cup nS_\beta cl(B) \subseteq nS_\beta cl\ (A \cup B)$.

iii. $nS_\beta cl(A \cap B) \subseteq nS_\beta cl(A) \cap nS_\beta cl(B)$.

Proof. Straightforward.

xl. The inclusion in (ii and iii) of the above theorem cannot be replaced by quality in general, as it is shown in the following example.

xli. **Example 3.20.** Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then $\tau_R(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, b, c\}\}$, $nS_\beta O(U, X) = \{\phi, U, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}\}$ and $nS_\beta C(U, X) = \{\phi, U, \{b, c, d\}, \{a, d\}, \{b, c\}, \{d\}, \{a\}\}$. For part ($ii$), take $F = \{a, d\}$ and $E = \{a, b\}$, then $nS_\beta cl(F \cap E) = nS_\beta cl(\{a\}) = \{a\}$, but $nS_\beta cl(F) \cap nS_\beta cl(E) = \{a, d\} \cap U = \{a, d\}$. Therefore, $nS_\beta cl(F \cap E) \neq S_\beta cl(F) \cap nS_\beta cl(E)$. For part ($iii$), take $F = \{b, c\}$ and $E = \{a\}$, then $nS_\beta cl(\{b, c\} \cup \{a\}) = U$, but $nS_\beta cl(\{b, c\}) \cup nS_\beta cl(\{a\}) = \{a, b, c\}$. Therefore, $nS_\beta cl(F) \cup nS_\beta cl(E) \neq nS_\beta cl\ (F \cup E)$.

xlii. **Theorem 3.21.** Let A be any subset of a nano topological space $(U, \tau_R(X))$, then the following statements are true:

i. $nS_\beta cl(\phi) = \phi$ and $nS_\beta cl(U) = U$.
ii. $A \subseteq nS_\beta cl(A)$.
iii. $A \in nS_\beta C(U, X)$ if and only if $A = nS_\beta cl(A)$.
iv. $nS_\beta cl\left(nS_\beta cl(A)\right) = nS_\beta cl(A)$

Proof. Straightforward.

xliii. **Corollary 3.22.** Let $(U, \tau_R(X))$ be a nano topological space when $U_R(X) = U$ and $L_R(X) = \phi$, then for any non-empty subset A of U, $nS_\beta cl(A) = U$

Proof. Follows from Theorem 2.8.

xliv. **Corollary 3.23.** Let $(U, \tau_R(X))$ be a nano topological space when $U_R(X) = U$ and $L_R(X) \neq \phi$, then for any non-empty subset A of U:

$$nS_\beta cl(A) = \begin{cases} [L_R(X)]^c, & if\ A \subseteq L_R(X) \\ [B_R(X)]^c, & if\ A \subseteq B_R(X) \\ U, & otherwise \end{cases}$$

xlv.
Proof. By Theorem 2.9. we have $\phi, U, L_R(X)$ and $B_R(X)$ are the only nS_β-open sets in U, then $\phi, U, [L_R(X)]^c$ and $[B_R(X)]^c$ are the only nS_β-closed sets in U, so we get the result.

xlvi. **Corollary 3.24.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) = \{x\}$, $x \in U$, then for any non-empty subset A of U, $nS_\beta cl(A) = U$

xlvii. **Proof.** Follows from Theorem 2.10.

xlviii. **Corollary 3.25.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U, then for any non-empty subset A of U:

$$nS_\beta cl(A) = \begin{cases} [U_R(X)]^c, & if\ A = [U_R(X)]^c \\ A, & if\ A \subset [U_R(X)]^c \\ U, & otherwise \end{cases}$$

xlix.
Proof. By Theorem 2.11, the only nS_β-closed sets are ϕ and those subsets A for which $A \subseteq [U_R(X)]^c$. If $A = [U_R(X)]^c$, then $nS_\beta cl([U_R(X)]^c) = [U_R(X)]^c$. If $A \subset [U_R(X)]^c$, then A is nS_β-closed in U, hence $nS_\beta cl(A) = A$. If $[U_R(X)]^c \subset A$, since $[U_R(X)]^c$ is the largest nS_β-closed set in U, hence $nS_\beta cl(A) = U$.

l. **Corollary 3.26.** Let $(U, \tau_R(X))$ be a nano topological space. If $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U, then for any non-empty subset A of U:

$$nS_\beta cl(A) = \begin{cases} [U_R(X)]^c, & if\ A = [U_R(X)]^c \\ A, & if\ A \subset [U_R(X)]^c \\ U, & otherwise \end{cases}$$

li.
Proof. The proof is similar to Corollary 3.25.

4. NANO S_β-CONTINUOUS FUNCTIONS

lii. **Definition 4.1.** A function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be nS_β-continuous at a point $x \in U$, if for each nano open set B in V containing $f(x)$, there exists an nS_β-open set A in U containing x such that $f(A) \subseteq B$. If f is nS_β-continuous at every point x of U, then it is called nS_β-continuous.

liii. **Definition 4.2.** A function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be nS_β-irresolute if the inverse image of every nS_β-open set is nS_β-open.

liv. **Theorem 4.3.** Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be a function, then the following statements are equivalent:

1. f is nS_β-continuous function.
2. The inverse image of every nano open set B in V is nS_β-open set in U.
3. The inverse image of every nano closed set F in V is nS_β-closed set in U.
4. For each $A \subseteq U, f\left(nS_\beta cl(A)\right) \subseteq ncl(f(A))$.
5. For each $A \subseteq U, nint(f(A)) \subseteq f\left(nS_\beta int(A)\right)$.
6. For each $B \subseteq V, nS_\beta cl(f^{-1}(B)) \subseteq f^{-1}(ncl(B))$.
7. For each $B \subseteq V, f^{-1}(nint(B)) \subseteq nS_\beta int(f^{-1}(B))$.

Proof. Straightforward.

lv. The equality in parts $(4, 5, 6\ and\ 7)$ in the above theorem does not hold in general, as it is shown in the following example.

lvi. **Example 4.4.** Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, c\}$. Then $\tau_R(X) = \{\phi, U, \{c\}, \{a, b, c\}, \{a, b\}\}$ and $nS_\beta O(X) = \{\phi, U, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$. Let $V = \{x, y, w, z\}$ with $V/R' = \{\{x\}, \{z\}, \{y, w\}\}$ and $Y = \{x, z\}$. Then $\tau_{R'}(Y) = \{\phi, V, \{x\}, \{y, w\}, \{x, y, w\}\}$. Define $f: U \to V$ as $f(a) = y, f(b) = w, f(c) = x$ and $f(d) = z$, then f is nS_β-continuous function.

i. For part (4), take $A = \{a\}$, then $\{y, w\} = f(nS_\beta clA) \not\supseteq ncl(f(A)) = \{y, w, z\}$.

ii. For part (5), take $A = \{c, d\}$, then $\{x\} = nint(f(A)) \not\supseteq f(nS_\beta int(A)) = \{x, z\}$.

iii. For part (6), take $F = \{x\}$, then, $\{c\} = nS_\beta cl(f^{-1}(B)) \not\supseteq f^{-1}(ncl(B)) = \{c, d\}$.

iv. For part (7), take $F = \{y, w, z\}$, then $\{a, b\} = f^{-1}(int(B)) \not\supseteq nS_\beta int(f^{-1}(B)) = \{a, b, d\}$.

Remark 4.5. Nano continuous function and nS_β-continuous function are independent in general, as it is shown by the following example.

lvii.**Example 4.6.** Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$, then $\tau_R(X) = \{\phi, U, \{a\}\}$, then $nS_\beta O(X) = \{\phi, U\}$. Define the identity function $f: (U, \tau_R(X)) \to (U, \tau_R(X))$, consider $f^{-1}(\{a\}) = \{a\} \in \tau_R(X)$ but $\{a\} \notin nS_\beta O(X)$. Therefore, f is nano continuous but not nS_β-continuous function. Also, let $V = \{a, b, c, d\}$ with $V/R = \{\{a, b\}, \{c, d\}\}$ and $X = \{a, b\}$, then $\tau_R(X) = \{\phi, V, \{a, b\}\}$ and $nS_\beta O(V, X) = \{\phi, V, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Define the function $f: (V, \tau_R(X)) \to (V, \tau_R(X))$ by $f(a) = a, f(b) = a, f(c) = b$ and $f(d) = d$, then $f^{-1}(\{a, b\}) = \{a, b, c\} \notin \tau_R(X)$. Therefore, f is not nano continuous but nS_β-continuous function.

lviii.**Proposition 4.7.** Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be a function, then the following statements are true:

lix.Every nS_β-continuous function is nS-continuous.

lx.Every nS_β-continuous function is nS-continuous.

Proof. Obvious.

lxi.Example 4.6 shows that the converse of above proposition is not true in general.

lxii.**Proposition 4.8.** If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is nS-continuous function and U is locally indiscrete, then f is nS_β-continuous.

Proof. Suppose that f is nS-continuous function and B be any nano open set in V. Then $f^{-1}(B)$ is nS-open subset of U, since U is locally indiscrete, then by Proposition , $f^{-1}(B) \in nS_\beta O(U, X)$. Hence f is nS_β-continuous function.

lxiii.**Theorem 4.9.** A function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is nS_β-continuous if and only if f is nS-continuous and for each $x \in U$ and each nano open set B of V containing $f(x)$, there exists an $n\beta$-closed set F in U containing x such that $f(F) \subseteq B$.

Proof. Let B be any nano open set in V. Since f is nS-continuous, then $f^{-1}(B)$ is nS-open set in U. Let $x \in f^{-1}(B)$, so $f(x) \in B$. By assumption, there exists a $n\beta$-closed set F of U such that $f(F) \subseteq B$. Which it implies that $x \in F \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B)$ is a nS_β-open set in U. Hence f is nS_β-continuous function.
Conversely, follows from Proposition 4.7.

lxiv.**Theorem 4.10.** Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be an nS_β-continuous function and U is extremely disconnected, then f is $n\alpha$-continuous.

Proof. Let f be nS_β-continuous function and A be any nano open subset of V, since f is nS_β-continuos, then $f^{-1}(A) \in nS_\beta O(U,X)$, by Proposition , $f^{-1}(A) \in nSO(U,X)$ and then $f^{-1}(A) = \cup_{\lambda\in\Delta} F_\lambda$, where $F_\lambda \in n\beta C(U,X)$ for each $\lambda \in \Delta$, but since U is extramelly disconnected, so $f^{-1}(A) \subseteq ncl\left(nint\left(f^{-1}(A)\right)\right) = nint\left(ncl\left(f^{-1}(A)\right)\right)$. Therefore, $f^{-1}(A) \in n\alpha O(U,X)$. Thus, f is $n\alpha$-continuous.

lxv.**Theorem 4.11.** If the function $f: \left(U, \tau_R(X)\right) \to \left(V, \tau_{R'}(Y)\right)$ is $n\alpha$-continuous function and U is locally indiscrete, then f is nS_β-continuous.

lxvi.**Proof.** Let f is $n\alpha$-continuous function, then the $f^{-1}(G)$ is $n\alpha$-open in U for every nano open set G in V. Since $f^{-1}(G)$ is also nS-open and U is locally indiscrete, by Proposition 4.8, f is a nS_β-continuous.

lxvii.**Theorem 4.12.** The identity function $f: \left(U, \tau_R(X)\right) \to \left(U, \tau_R(X)\right)$ is nS_β-continuous function if $\left(U, \tau_R(X)\right)$ is extremely disconnected.

Proof. Obvious.

Theorem 4.13. A function $f: \left(U, \tau_R(X)\right) \to \left(V, \tau_{R'}(Y)\right)$ is nS_β-continuous function if $U_{R'}(Y) = V$ and $L_{R'}(Y) = \phi$.

Proof. $\tau_{R'}(Y) = nS_\beta(V,Y) = \{\phi, U\}$. Then $f^{-1}(\phi) = \phi$ and $f^{-1}(U) = U$. Hence f is nS_β-continuous function.

Theorem 4.14. The identity function $f: \left(U, \tau_R(X)\right) \to \left(U, \tau_R(X)\right)$ is nS_β-continuous function if:

i. $U_R(X) = U$ and $L_R(X) = \phi$.

ii. $U_R(X) = U$ and $L_R(X) \neq \phi$.

iii. $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.

iv. $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.

v. $U_R(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R(X) \neq \phi$.

Proof.

lxviii.Since $\tau_R(X) = \{\phi, U\}$, then $nS_\beta O(U,X) = \{\phi, U\}$. Hence f is nS_β-continuous function.

lxix.Since $\tau_R(X) = \tau_R^{S_\beta}(X) = \{\phi, U, L_R(X), B_R(X)\}$. Hence f is nS_β-continuous function.

lxx.Since $\tau_R(X) = \{\phi, U, U_R(X)\}$, then $U_R(X) \in nS_\beta O(U,X)$. Hence f is nS_β-continuous function.

lxxi.Since $\tau_R(X) = \{\phi, U, U_R(X)\}$, then $U_R(X) \in nS_\beta O(U,X)$. Hence f is nS_β-continuous function.

lxxii.Since $\tau_R(X) = \{\phi, U, L_R(X), B_R(X), U_R(X)\}$, then $\tau_R(X) \subset nS_\beta O(U,X)$. Hence f is nS_β-continuous function.

lxxiii.**Remark 4.15.** The composition of two nS_β-continuous functions need to be nS_β-continuous function in general, as it is shown by the following example.

lxxiv.**Example 4.16.** Let $U = \{a, b, c, d, e\}$ with

$U/R = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{a, c, d\}$.

Also, let $U/R' = \{\{a\}, \{b, c\}, \{d\}, \{e\}\}$ and

$Y = \{a, b, d\}$. Then

$\tau_R(X) = \{\phi, U, \{a, b, c, d\}, \{c, d\}, \{a, b\}\}$ and

$\tau_{R'}(Y) = \{\phi, U, \{a, b, c, d\}, \{a, d\}, \{b, c\}\}$. Then

lxxv.$nS_\beta O(U, X) = \{\phi, U, \{a, b, c, d\}, \{c, d\}, \{a, b\}, \{c, d, e\}, \{a, b, e\}\}$ and

$nS_\beta O(U, Y) = \{\phi, U, \{a, b, c, d\}, \{c, d\}, \{a, b\}, \{a, d, e\}, \{b, c, e\}\}$

lxxvi.. Define the function $f: (U, \tau_R(X)) \to$

$(U, \tau_{R'}(Y))$ by

$f(a) = b, f(b) = c, f(c) = a, f(d) = d$, and

$f(e) = e$. Define $g: (U, \tau_{R'}(Y)) \to (U, \tau_R(X))$

by $g(a) = d, g(b) = a, g(c) = b, g(d) = c$,

and $g(e) = a$. It is clear f and g are nS_β-continuous functions. Then

$h = fog: (U, \tau_R(X)) \to (U, \tau_R(X))$ is

$h(a) = d, h(b) = b, h(c) = c, h(d) = a$ and

$h(e) = b$. Since $\{c, d\} \in \tau_R(X)$ but

$h^{-1}(\{c, d\}) = \{a, c\} \notin nS_\beta O(U, X)$. Hence, the

composition of nS_β-continuous functions need

to be nS_β-continuous.

Theorem 4.17. Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be nS_β-continuous and $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ be a nano continuous functions. Then the composition functions $fog: (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is nS_β-continuous.

Proof. Let H be any nano open subset of $(W, \tau_{R''}(Z))$. Since g is nano continuous, then $g^{-1}(H)$ is a nano open subset of $(V, \tau_{R'}(Y))$. Since f is nS_β-continuous, then by Theorem 4.3., $(fog)^{-1}(H) = f^{-1}(g^{-1}(H))$ is nS_β-open subset in $(U, \tau_R(X))$. Therefore fog is nS_β-continuous function.

5. CONCLUSION

In this paper, we have introduced the nano nS_β-operators by using nano S_β-open sets such as nano S_β-interior and nano S_β-closure with their properties and they are clarified by $U_R(X)$, $L_R(X)$ and $B_R(X)$ approximations. Also, the concept of nano S_β-continuity have introduced and the relationship among some types of nano continuous function in nano topological spaces are considered such as we can see that nano continuous function and nS_β-continuous function are independent in general. Also, we have shown that every nS_β-continuous function is nS-continuous but the converse may not be true. Furthermore, it is shown that the composition of two nS_β-continuous functions need to be nS_β-continuous function in general.

6. REFERENCES

Pawlak, Z. (1982). Rough sets. *International journal of computer & information sciences*, *11*(5), 341-356.

Thivagar, M. L., & Richard, C. (2013). On nano forms of weakly open sets. *International journal of mathematics and statistics invention*, *1*(1), 31-37.

Revathy, A., & Ilango, G. (2015). On nano β-open sets. *Int. J. Eng. Contemp. Math. Sci*, *1*(2), 1-6.

Pirbal, O. T, & Ahmed, N. K. (2022). On nano $S\beta$-open sets in nano topological spaces, General letters in mathematics, 12 (1), 23-30.

Thivagar, M. L., & Richard, C. (2013). On nano continuity. *Mathematical theory and modeling*, *3*(7), 32-37.

Nasef, A. A., Aggour, A. I., & Darwesh, S. M. (2016). On some classes of nearly open sets in nano topological spaces. *Journal of the Egyptian Mathematical Society*, *24*(4), 585-589.

Thivagar, M. L., & Richard, C. (2013). Note on nano topological spaces, communicated.