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DETOUR POLYNOMIALS OF COG-COMPLETE BIPARTITE GRAPH

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ABSTRACT

The detour distance is a topological concept of graph theory, denoted by $D(u,v)$ and defined as the length of a longest (u,v) – path in a connected graph G, where the vertices u and v belonged to the vertex set $V(G)$. In this paper, we find the detour polynomial, detour index for cog-complete-bipartite graphs, also, some special cases were taken for cog-complete bipartite graph to show the gear so that we can determine detour polynomial and detour index for any order.

KEYWORD : detour distance, detour polynomial, cog-complete-bipartite graph, fan graph.

1. INTRODUCTION

Let G be a connected graph of order p and size q. Then, for each unordered pairs u,v of vertices of G, the **distance $d(u,v)$** is the minimum of the lengths of all $u-v$ paths in G, [6,10] . The **detour distance** between two distinct vertices u and v in a connected graph G is the maximum of the lengths of all $u-v$ paths in G (see [4]). Moreover, $D(u,u)=0$, for each $u\in V(G)$, if uv is a bridge of G then $D(u,v)=1$, and for every vertex u and $v\in V(G)$ $D(u,v)=d(u,v)$ if and only if G is a tree. Also, it is clear that $D(u,v)=p-1$ if and only if G contains a Hamiltonian $u-v$ path. The **detour eccentricity** denoted by $e_D(v)$ of a vertex v is the maximum detour distance from v to all other vertices in G, the parameters $\delta_D(G)$ or $diam_D(G)$ denote the **detour diameter** of G which is defined as the maximum detour eccentricity among all vertices in G. The vertex v is called **a peripheral** vertex of G if $e_D(v)=\delta_D(G)$ and the set of all peripheral vertices of G is **the peripheral** of G and it is denoted $P_D(G)$. The **detour radius** $rad_D(G)$ of G is the minimum detour eccentricity among all vertices in G. Obviously $e(v)\le e_D(v)$ for every vertex v in G, since $d(u,v)\le D(u,v)$, for u and v in G. Therefore, $diam(G)\le diam_D(G)$ and $rad(G)\le rad_D(G)$. If a vertex v of G satisfy the property that it's eccentricity is $rad_D(G)$ then v is said to be The **detour central** vertex of G. The **detour central** of G, $C_D(G)$ is the set of all center vertices of G, $m_D(G)$ is the **minimum detour distance** defined by $m_D(G)=\min\{D(u,v):\{u,v\}\subseteq V(G)\}$.

Werefer the reader to [5,7,8,9],
for details of detour polynomial, detour index and some properties.

The **detour index $dd(G)$** of G is defined as:

$$dd(G)=\sum_{\{u,v\}\in V(G)} D(u,v) \quad \dots (1.1)$$

Thedistance polynomial [13]ofaconnectedgraph G basedondetourdistanceis called **detour polynomial $D(G;x)$**, which is defined as follows:

$$D(G;x)=\sum_{\{u,v\}} x^{D(u,v)}. \quad \dots (1.2)$$

Where the summation is taken over all unordered pairs of distinct vertices u and v of G .It is clear that

$$dd(G)=\frac{d}{dx}D(G;x)|_{x=1}. \quad \dots. (1.3)$$

Let $C_D(G,k)$ is the number of unordered pairs u and v such that $D(u,v)=k$, then the **detour polynomial** $D(G;x)$ of a connected graph G, is also defined by:

$$D(G;x)=\sum_{k\ge1} C_D(G,k)x^k. \quad \dots (1.4)$$

The detour polynomial of a vertex v in G is define as:

$$D(v,G;x)=\sum_{k\ge1}^{e_D(v)} C_D(v,G,k)x^k, \quad \dots (1.5)$$

where $C_D(v,G,k)$ be the number of vertices u, $(u\ne v)$ such that $D(u,v)=k$.

It is clear that

$$D(G;x)=\frac{1}{2}\sum_{v\in V(G)} D(v,G;x). \quad \dots (1.6)$$

Observe that $D(v,G;1)=p-1$.

Chartrand, Escuardo and Zhang introduced the concept of detour distance, but the concept of detour distance polynomial of a connected graph G was introduced, Mohammed in [13] found polynomials of detour for special graphs and

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operations defined on graphs , some work has been done on detour indices. Several authors had obtained detour number, detour polynomials and detour indices for many structures graphs [1, 3, 11, 12].

2. DETOUR POLNOMIAL OF COG-COMPLETE-BIPARTITE GRAPH

Definition:[2] A **cog-complete-bipartite graph $K_{n,m}^c$** is the graph constructed from a complete-bipartite graph $K_{n,m}$, $n,m \geq 2$ of vertex sets $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_m\}$, by adding $(m+n-2)$ vertices $W = \{w_1, w_2, \dots, w_{n-1}\}$ *and* $Y = \{y_1, y_2, \dots, y_{m-1}\}$ to the graph $K_{n,m}$ with $2(m+n-2)$ edges $\{v_i w_i, v_{i+1} w_i : i = 1,2,\dots,n-1\} \cup \{u_j y_j, u_{j+1} y_j : j = 1,2,\dots,m-1\}$. See Figure 2.1.

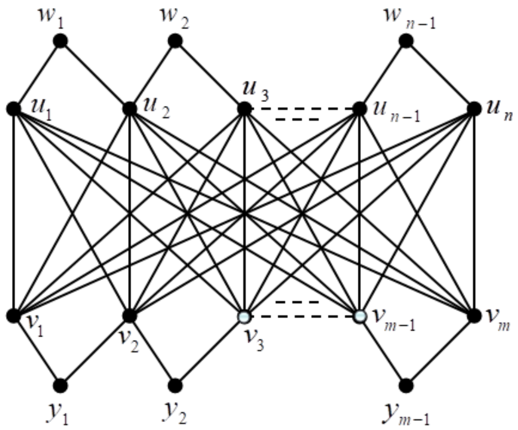

Fig. (2.1): Cog- Complete-Bipartite Graph $K_{n,m}^c$.

Proposition 2.1: Let $K_{n,m}^c$ be a cog – complete – bipartite graph, $= \{u_1, u_2, \dots, u_n\}$, $V = \{v_1, v_2, \dots, v_m\}$, $W = \{w_1, w_2, \dots, w_{n-1}\}$ *and* $Y = \{y_1, y_2, \dots, y_{m-1}\}$. Then for all $n, m \geq 4$, we have:

1. If $u_i, u_j \in U$, then:

$$D(u_i, u_j) = \begin{cases} 2n+2m-6, & if\ |i-j| \neq 1,\ for\ i,j = 2,\dots,n-1, i \neq j, \\ 2n+2m-4, & otherwise. \end{cases}$$

2. If $v_i, v_j \in V$, then:

$$D(v_i, v_j) = \begin{cases} 2n+2m-6, & if\ |i-j| \neq 1,\ for\ i,j = 2,\dots,m-1, i \neq j, \\ 2n+2m-4, & otherwise. \end{cases}$$

3. If $w, w' \in W$, then:
$D(w, w') = 2n+2m-4$.

4. If $y, y' \in Y$, then:
$D(y, y') = 2n+2m-4$.

5. If $u \in U$ and $w \in W$, then:

$$D(u, w) = \begin{cases} 2n+2m-3, & if\ u = u_1 or\ u = u_n\ or\ uw \in E(K_{n,m}^c), u \neq u_1, u_n, \\ 2n+2m-5, & otherwise. \end{cases}$$

6. If $v \in V$ and $y \in Y$, then:

$$D(v, y) = \begin{cases} 2n+2m-3, & if\ v = v_1 or\ v = v_m\ or\ vy \in E(K_{n,m}^c), v \neq v_1, v_m, \\ 2n+2m-5, & otherwise. \end{cases}$$

7. If $u_i \in U$ and $v_j \in V$, then:

$$D(u_i, v_j) = \begin{cases} 2n+2m-3, & when\ i = 1, n\ and\ j = 1, m, \\ 2n+2m-5, & otherwise. \end{cases}$$

8. If $u \in U$, $v \in V$, $w \in W$ and $y \in Y$, then:
$D(u, y) = D(v, w, K_{n,m}^c) = 2n+2m-4$ and $D(w, y) = 2n+2m-3$.

The detour polynomial of cog-bipartite-complete graph $K_{n,m}^c$ is sought out in the next theorem:

Theorem 2.2: For $n, m \geq 4$, then

$$D(K_{n,m}^c; x) = \{mn + 3(n+m) - 7\}x^{2n+2m-3} + \frac{1}{2}\{4(mn-5) + n(n+1) + m(m+1)\}x^{2n+2m-4} + \{nm + n(n-5) + m(m-5) + 8\}x^{2n+2m-5} + \frac{1}{2}\{n(n-7) + m(m-7) + 24\}x^{2n+2m-6}. \quad \ldots (1.1)$$

Proof:

From definition and Proposition 2.1, we get,

1. $\sum_{\{u,v\}\subseteq U} D(u, v, K_{n,m}^c; x) = (3n-6)x^{2n+2m-4} + \{\binom{n}{2} - (3n-6)\}x^{2n+2m-6}$.

2. $\sum_{\{u,v\}\subseteq V} D(u, v, K_{n,m}^c; x) = (3m-6)x^{2n+2m-4} + \{\binom{m}{2} - (3m-6)\}x^{2n+2m-6}$.

3. $\sum_{\{u,v\}\subseteq W} D(u, v, K_{n,m}^c; x) = \binom{n-1}{2} x^{2n+2m-4}$.

4. $\sum_{\{u,v\}\subseteq Y} D(u, v, K_{n,m}^c; x) = \binom{m-1}{2} x^{2n+2m-4}$.

5. $\sum_{\substack{u\in U\\ w\in W}} D(u, w, K_{n,m}^c; x) = 2(2n-3)x^{2n+2m-3} + \{n(n-1) - 2(2n-3)\}x^{2n+2m-5}$.

6. $\sum_{\substack{v\in V\\ y\in Y}} D(v, y, K_{n,m}^c; x) = 2(2m-3)x^{2n+2m-3} + \{m(m-1) - 2(2m-3)\}x^{2n+2m-5}$.

7. $\sum_{\substack{u\in U\\ v\in V}} D(u, v, K_{n,m}^c; x) = 4x^{2n+2m-3} + (nm-4)x^{2n+2m-5}$.

8. $\sum_{\substack{u\in U\\ y\in Y}} D(u, y, K_{n,m}^c; x) = n(m-1)x^{2n+2m-4}$,

$\sum_{\substack{v\in V\\ w\in W}} D(v, w, K_{n,m}^c; x) = m(n-1)x^{2n+2m-4}$,

$\sum_{\substack{y\in Y\\ w\in W}} D(y, w, K_{n,m}^c; x) = (n-1)(m-1)x^{2n+2m-3}$.

Since,

$$D(K_{n,m}^c; x) = \sum D(u, K_{n,m}^c; x) + \sum D(v, K_{n,m}^c; x) + \sum D(w, K_{n,m}^c; x) + \sum D(y, K_{n,m}^c; x) + \sum D(u, w, K_{n,m}^c; x) + \sum D(v, y, K_{n,m}^c; x) + \sum D(u, v, K_{n,m}^c; x) + \sum D(u, y, K_{n,m}^c; x) + \sum D(v, w, K_{n,m}^c; x) + \sum D(w, y, K_{n,m}^c; x).$$

Hence,

$$\begin{aligned} D(K_{n,m}^c; x) &= \{2(2n-3) + 2(2m-3) + 4 + (n-1)(m-1)\}x^{2n+2m-3} \\ &+\{(3n-6) + (3m-6) + \tfrac{1}{2}(n-1)(n-2) + \tfrac{1}{2}(m-1)(m-2) \\ &+n(m-1) + m(n-1)\}x^{2n+2m-4} \\ &+\{n(n-1) - 2(2n-3) + m(m-1) - 2(2m-3) + nm - 4\}x^{2n+2m-5} \\ &+\{\tfrac{1}{2}n(n-1) - (3n-6) + \tfrac{1}{2}m(m-1) - (3m-6)\}x^{2n+2m-6}. \\ &= \{mn + 3(n+m) - 7\}x^{2n+2m-3} \\ &+ \tfrac{1}{2}\{4(mn-5) + n(n+1) + m(m+1)\}x^{2n+2m-4} \\ &+\{nm + n(n-5) + m(m-5) + 8\}x^{2n+2m-5} \\ &+ \tfrac{1}{2}\{n(n-7) + m(m-7) + 24\}x^{2n+2m-6}. \end{aligned}$$

■

Corollary 2.3: For $m \geq 4$, then

$$D(K_{3,m}^c; x) = 2(3m+1)x^{2m+3} + \frac{1}{2}(m^2 + 13m - 8)x^{2m+2} + (m(m-2) + 2)x^{2m+1} + \frac{1}{2}(m-3)(m-4)x^{2m}. \quad \ldots (2.2)$$

Proof:

To obtain the formula $D(K_{3,m}^c; x)$, we follow the same steps as the proof of Theorem 2.2 when $n = 3$.

■

Remark: $(K_{3,3}^c; x) = 20x^9 + 20x^8 + 5x^7$.

Corollary 2.4: Let $K_{2,m}^c$ be a cog – complete – bipartite graph, then for all $m \geq 4$, we have:

$$D(K_{2,m}^c; x) = (5m-1)x^{2m+1} + \frac{1}{2}(m^2 + 9m - 12)x^{2m} + (m-2)(m-1)x^{2m-1}$$

$+\frac{1}{2}(m-3)(m-4)x^{2m-2}$. … (2.3)

Proof:

Follows from theorem 2.2 when n=2:

1. $\sum_{\{u,v\}\subseteq U} D(u,v,K^c_{2,m};x) = x^{2m}$, where $U = \{u_1,u_2\}$.

2. $\sum_{\{u,v\}\subseteq V} D(u,v,K^c_{2,m};x) = 3(m-2)x^{2m} + \frac{1}{2}(m-3)(m-4)x^{2m-2}$.

3. $\sum_{\{u,v\}\subseteq Y} D(u,v,K^c_{2,m};x) = \frac{1}{2}(m-1)(m-2)x^{2m}$.

4. $\sum_{\substack{u\in U\\ w\in W}} D(u,w,K^c_{2,m};x) = 2x^{2m+1}$, where $W = \{w_1\}$.

5. $\sum_{\substack{v\in V\\ y\in Y}} D(v,y,K^c_{2,m};x) = 2(2m-3)x^{2m+1} + (m-2)(m-3)x^{2m-1}$.

6. $\sum_{\substack{u\in U\\ v\in V}} D(u,v,K^c_{2,m};x) = 4x^{2m+1} + 2(m-2)x^{2m-1}$.

7. $\sum_{\substack{u\in U\\ y\in Y}} D(u,y,K^c_{2,m};x) = 2(m-1)x^{2m}$,

$\sum_{\substack{v\in V\\ w\in W}} D(v,w,K^c_{2,m};x) = mx^{2m}$,

$\sum_{\substack{y\in Y\\ w\in W}} D(y,w,K^c_{2,m};x) = (m-1)x^{2m+1}$.

Since,
$$\begin{aligned}
D(K^c_{2,m};x) &= \sum D(u,K^c_{2,m};x) + \sum D(v,K^c_{2,m};x) \\
&\quad + \sum D(y,K^c_{2,m};x) + \sum D(u,w,K^c_{2,m};x) + \sum D(v,y,K^c_{2,m};x) \\
&\quad + \sum D(u,v,K^c_{2,m};x) + \sum D(u,y,K^c_{2,m};x) \\
&\quad + \sum D(v,w,K^c_{2,m};x) + \sum D(w,y,K^c_{2,m};x). \\
&= x^{2m} + 3(m-2)x^{2m} + \frac{1}{2}(m-3)(m-4)x^{2m-2} \\
&\quad + \frac{1}{2}(m-1)(m-2)x^{2m} + 2x^{2m+1} \\
&\quad + 2(2m-3)x^{2m+1} + (m-2)(m-3)x^{2m-1} + 4x^{2m+1} \\
&\quad + 2(m-2)x^{2m-1} + 2(m-1)x^{2m} + mx^{2m} + (m-1)x^{2m+1} \\
&= (5m-1)x^{2m+1} + \frac{1}{2}(m^2+9m-12)x^{2m} \\
&\quad + (m-2)(m-1)x^{2m-1} + \frac{1}{2}(m-3)(m-4)x^{2m-2}. \quad \blacksquare
\end{aligned}$$

Remark :

- $D(K^c_{2,3};x) = 14x^7 + 12x^6 + 2x^5$.
- $D(K^c_{2,2};x) = 9x^5 + 6x^4$.

3. DETOUR INDEX OF COG- COMPLETE-BIPARTITE GRAPH:

Theorem 3.1: For $n,m \geq 4$, then

$D(K^c_{n,m}) = 4(m^3+n^3) - 20(m^2+n^2) + 41(m+n) + 12mn(m+n-3) - 51$. … (3.1)

Proof:

By taking the derivatives of equation(2.1), when x=1 of the detour polynomial of cog complete bipartite graph, then

$$\begin{aligned}
D(K^c_{n,m}) &= \frac{d}{dx}(D(K^c_{n,m}:x))|_{x=1} \\
&= \frac{d}{dx}(\{mn+3(n+m)-7\}x^{2n+2m-3} \\
&\quad + \frac{1}{2}\{4(mn-5)+n(n+1)+m(m+1)\}x^{2n+2m-4} \\
&\quad + \{nm+n(n-5)+m(m-5)+8\}x^{2n+2m-5} \\
&\quad + \frac{1}{2}\{n(n-7)+m(m-7)+24\}x^{2n+2m-6}.\ )|_{x=1} \\
&= \{mn+3(n+m)-7\}(2n+2m-3) \\
&\quad + \frac{1}{2}\{4(mn-5)+n(n+1)+m(m+1)\}(2n+2m-4) \\
&\quad + \{nm+n(n-5)+m(m-5)+8\}(2n+2m-5)
\end{aligned}$$

$$+\frac{1}{2}\{n(n-7)+m(m-7)+24\}(2n+2m-6)$$
$$=2m^2n+6m^2+2mn^2+9mn-23m+6n^2-23n+21$$
$$+m^3+5m^2n-m^2+5mn^2-6mn-22m+n^3-n^2-22n+40$$
$$+2m^3+4m^2n-15m^2+4mn^2-25mn+41m+2n^3-15n^2+41n-40$$
$$+m^3+m^2n-10m^2+mn^2-14mn+45m+n^3-10n^2+45n-72$$
$$=4m^3+12m^2n-20m^2+12m\,n^2-36mn+41m+4n^3-20n^2$$
$$+41n-51$$
$$=4(m^3+n^3)-20(m^2+n^2)+41(m+n)+12mn(m+n-3)-51. \quad ■$$

Corollary 3.2: For $m \geq 4$, then

$D(K^c_{3,m}) = m(4m^2+16m+41)$. … (3.2)

Proof:

Only put n=3 in Theorem 3.1.■

Corollary 3.3: For $m \geq 4$, then

$D(K^c_{2,m}) = 4m^3+4m^2+17m-15$. ■

Some Properties of $K^c_{n,m}$, for all $n, m \geq 3$, we have:

- **Order and size:**The order of $K^c_{n,m}$ is $2(n+m-1)$ and the size is $mn+2(n+m-2)$.
- **Diameter and radius of detour distance:** $\delta_D(K^c_{n,m})$=$r_D(K^c_{n,m}) = 2(n+m)-3$.
- **Minimum detour distance :** $m_D(K^c_{n,m}) = 2(n+m)-6, n, m \geq 5$.
- **Degree vertices:** The vertices u_i and v_j have degree $m+1$ and $n+1$ respectively, for $i=1,n$, $j=1,m$, the vertices u_i and v_j have degree $m+2$ and $n+2$ respectively, for $i=2,\dots,n-1$, $j=2,\dots,m-1$, and added vertices w_i and y_j have degree two, for all $i=1,2,\dots,n-1$, $j=1,2,\dots,m-1$.
- **Detour peripheral of K^c_n:** $P_D(K^c_{n,m}) = V(K^c_{n,m})$.
- **Detour center of K^c_n:** $C_D(K^c_{n,m}) = V(K^c_{n,m})$.

4. CONCLUSION

In this paper, we conclude that we will use a known methods to evaluate the detour polynomial and index of such graphs. And also note that if the number of vertices of the graph G is separate for each cases, it will be easier to calculate the detour polynomial of the graphs.

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