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ON TOPOLOGICALLY ζ-TRANSITIVE MAPS

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ABSTRACT

In this paper, we describe and investigate some of properties of a new type of topologically transitive maps called topologically ζ-transitive in a topological space (X, τ) via the concept of ζ-open sets. Some properties of these maps are given and relation among topological transitivity, topological α-transitivity and topological ζ-transitivity are discussed.

KEY WORDS AND PHRASES: ζ-open set, ζ-dense, topologically ζ-transitive, ζ-irresolute map, ζ-transitive point.

1. INTRODUCTION

Over the last four decades, the field of mathematics known as dynamical systems has seen great interest and expansion. A dynamical system is a sort of function that is used to model processes that change over time. Fluid mechanics, population expansion, particle dynamics, and a variety of other situations where a physical system evolves through time are examples of such processes. Several types of dynamical systems can be found such as transitivity, strong transitivity, exact and mixing transitivity and most of them are defined for topological and metric spaces, see [1], [2], [3] and [8].

Velicko [13] introduced the concepts of θ-open sets and θ-closure in topological spaces in order to explore the class of H-closed spaces. Jafari [6] has also investigated a number of new and fascinating characteristics about these sets and introduced the concept of quasi θ-continuous functions. In 1963, Levine [9] introduced the concept of semi-open sets in topological spaces. Several authors defined and analyzed stronger and weaker versions of topological concepts utilizing this notion. Njastad [12], introduced the concept of α-open set in topological spaces, and in 1971, Crossley and Hildebrand [5], defined the semi-closure of a set A in a topological space which is denoted by $scl(A)$ as an intersection of all semi-closed sets containing the set A and stated some its properties. Maheshwari and U. Tapi [15] introduced the concept of feebly open set, in a topological space (X, τ). Murad in [14], the concept of topologically θ-transitive and θ-minimal maps are defined. In [11], the notion of topologically α-transitive and α-minimal maps are defined. Hasan in [4], initiated the notion of ζ-open sets and several topological concepts were discussed by utilizing this set. In 2018, Ameen and Asaad, proved that α-open set and feebly open set are equivalent see [16].

In this paper, we introduce a new type of topologically transitive maps called topologically ζ-transitive maps by using the concept of ζ-open sets. Relations among these maps are investigated. Also, some properties of these maps are given by means of orbits and strong orbits of points.

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2. PRELIMINARIES

By a space X or (X, τ), we mean a topological space and if $A \subseteq X$, then the interior and the closure of A is denoted by $int(A)$ and $cl(A)$, respectively. In this section we give necessary definitions and results, which can be used in the sequel.

Definition 2.1. A subset A of a topological space (X, τ) is said to be: a semi- open [9] (resp., α-open [12]) set, if $A \subseteq cl(int(A))$ (resp., $A \subseteq int(cl(int(A)))$). The complement of a semi-open (resp., α-open) set is called semi-closed (resp., α-closed). Also, A is semi-closed (resp., α-closed) if $int(cl(A)) \subseteq A$) (resp., $cl(int(cl(A))) \subseteq A$).

Definition 2.2. [8] Let A be a subset of a topological space (X, τ), then semi-closure of A,

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defined as the intersection of all semi-closed sets which containing A, and is denoted by $scl(A)$.

Definition 2.3. [13] A subset A of a space X is said to be θ-open set if for every $x \in A$, there exists an open set U such that $x \in U \subseteq cl(U) \subseteq A$ and the family of all θ-open sets in X is denoted by $\theta O(X)$.

Definition 2.4. [4] The open set U of a space X is called ζ-open if for each $x \in U$ there exists a semi-closed set F such that $x \in F \subseteq U$. The family of all ζ-open sets of X is denoted by $\zeta O(X)$.

The complement of each ζ-open set is called ζ-closed and the family of all ζ-closed sets of a space X is denoted by $\zeta C(X)$.

Lemma 2.5. [4] For any subset A of a space (X, τ). If $A \in \theta O(X)$, then $A \in \zeta O(X)$.

Definition 2.6. [4] A point $x \in X$ is said to be a ζ-interior point of a subset A if there exists an ζ-open set U such that $x \in U \subseteq A$. The set of all ζ-interior points of A is denoted by $\zeta int(A)$.

Definition 2.7. [4] The intersection of all ζ-closed sets containing a set A is called the ζ-closure of A, which is denoted by $\zeta cl(A)$.

Definition 2.8. [4] Let (X, τ) be a topological space. A subset A of X is called:

1. ζ-dense in X if $\zeta cl(A) = X$.
2. nowhere ζ-dense, if $\zeta int(\zeta cl(A)) = \phi$.

Definition 2.9. A function $f: X \to X$ is called:

1. ζ-irresolute [4], if the inverse image of every ζ-open set is ζ-open.
2. ζ-open [4], if the image of every ζ-open set is ζ-open. A bijection map, which is both ζ-irresolute and ζ-open is called ζ-homeomorphism.
3. α-irresolute [10], if the inverse image of every α-open set is α-open.
4. θ-irresolute [14], if the inverse image of every θ-open set is θ-open.

Lemma 2.10. Let A be an ζ-dense subset of X, then $A \cap U \neq \phi$, for each $U \in \zeta O(X)$.

Proof. Let A be an ζ-dense subset in X, then by definition, $\zeta cl(A) = X$, and let U be a non-empty ζ-open set in X. Suppose that $A \cap U = \phi$. Therefore, $E = U^c$ is ζ-closed, where E is the complement of ζ-open and $A \subseteq U^c = E$. So, $\zeta cl(A) \subseteq \zeta cl(E)$, but $\zeta cl(A) = X$, so $X \subseteq E$, this contradiction $A \cap U \neq \phi$. □

Definition 2.11. [7] Let (X, τ) be a topological space and $f: X \to X$ a continuous map, then the pair (X, f) is called a dynamical system. The space X is called the phase space of the system (X, f).

Definition 2.12. [7] In a dynamical system (X, f), a subset A of X is called f-invariant if $f(A) \subseteq A$.

Definition 2.13. [3] The orbit of a point x in a system (X, f) is denoted by $O(x, f)$ and $O(x, f) = \{x, f(x), f^2(x), \ldots\}$ (by an orbit we mean a forward orbit even if f is a homeomorphism).

Definition 2.14. Let (X, τ) be a topological space and $f: X \to X$ a map, then f is called:

1. topologically transitive [3], if f is continuous and for every pair of open sets U and V in X, there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$.
2. topologically α-transitive [11], if f is α-irresolute and for every pair of α-open sets U and V in X, there exists a positive integer n such that $f^n(U) \cap (V) \neq \phi$.
3. topologically θ-transitive [14], if f is θ-irresolute and for every pair of θ-open sets U and V in X, there exists a positive integer n such that $f^n(U) \cap (V) \neq \phi$.

Lemma 2.15. [4] If (X, τ) is semi-T_1-space, then τ co-insides with $\zeta O(X)$.

Definition 2.16. [4] A space X is said to be:

1) ζT_1-space if every singleton set is ζ-closed.
2) ζ-regular if whenever A is ζ-closed in X and $x \notin A$, there exist disjoint ζ-open sets U and V with $x \in U$ and $A \subseteq V$

3. ζ-TRANSITIVE MAPS

In this section, we define ζ-transitive maps on a space (X, τ), and we investigate some of their properties and prove various conclusions using these new concepts.

Definition 3.1. Let (X, τ) be any topological space and $f: X \to X$ an ζ-irresolute map, then the pair (X, f) is called an ζ-dynamical system. In (X, f), the map f is called topologically ζ-transitive if for every pair of ζ-open sets U and V in X, there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$.

Example 3.2. Let $X = N$ with the topology having a subbase $\mathcal{S} = \{\{2n-1, 2n, 2n+1\}: n \in N\}$ and let $f: X \to X$ be defined as $f(2) = 3$, $f(3) = 2$ and $f(n) = n$ otherwise, then this map is not continuous, but it is ζ-irresolute. Also, we have $f^n(\{1,2,3\}) \cap \{5\} = \phi$ for each $n \in N$. Hence, f is not ζ-transitive.

Example 3.3. Let $X = R$ with the topology $\mathcal{T}$ defined as $G \in \mathcal{T}$ if and only if $0 \in G$. Consider the map $f: X \to X$ defined as $f(0) = 1$, $f(1) =$

0 and $f(x) = x$ otherwise, then this map is ζ-irresolute and ζ-transitive because the only non-empty ζ-open set is X.

Theorem 3.4. In a dynamical system (X, f). If f is ζ-irresolute and topologically transitive, then f is topologically ζ-transitive.

Proof. Let f be topologically transitive, ζ-irresolute. Let U, V be any two non-empty ζ-open sets in X, then U, V are open sets and by Definition 2.14, we obtain that there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$. Hence, f is topologically ζ-transitive. □

Corollary 3.5. In a semi-T_1-space X, a map $f : X \to X$ is topologically transitive if and only if it is ζ-transitive.

Proof. Let $f: X \to X$ be topologically transitive, then f is continuous and by Lemma 2.15, f is ζ-irresolute. Suppose that $U, V \in \zeta O(X)$, then U, V are open sets and by Definition 2.14, we obtain that there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$. Hence, f is ζ-transitive. □

The converse of Theorem 3.2 is not always true as shown in the following example:

Example 3.6. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}, \{a\}\}$. The family of all ζ-open sets is $\zeta O(X) = \{\phi, X, \{b\}, \{a, c\}\}$. Define the map $f : X \dashrightarrow X$ as follows: $f(a) = f(c) = b, f(b) = c$. Then, it can be easily seen that f is both ζ-irresolute and continuous. It is obvious that there is a positive integer n such that $f^n(\{a, c\}) \cap \{b\} \neq \phi$ and also $f^n(\{b\}) \cap \{a, c\} \neq \phi$. Hence, f is an ζ-transitive. Moreover, we have $f^n(\{b\}) \cap \{a\} = \phi$, for every positive integer n. Hence, f is not transitive.

Corollary 3.7. If f is ζ-irresolute and topologically α-transitive, then it is topologically ζ-transitive.

Proof. If U and V are ζ-open in X, then U, V are also α-open. Since f is topologically α-transitive, so there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$. Hence, f is topologically ζ-transitive. □

Corollary 3.8. If f is ζ-transitive, then for every pair of θ-open sets U and V there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$.

Proof. Let U and V be any θ-open sets in X, then U, V are also ζ-open. Since f is ζ-transitive, so there is a positive integer n such that $f^n(U) \cap (V) \neq \phi$. □

Proposition 3.9. If $f: X \to X$ is an ζ-irresolute function such that for each semi-closed sets A and B, there is a positive integer n in which $f^n(A) \cap B \neq \phi$, then f is topologically ζ-transitive.

Proof. Let U and V be any pair of ζ-open sets in X. Hence, by Definition 2.4, there exist semi-closed sets A and B such that $A \subseteq U$ and $\subseteq V$. Therefore, by hypothesis, there is a positive integer n such that $f^n(A) \cap (B) \neq \phi$. Consequently, $f^n(U) \cap (V) \neq \phi$ implies that f is topologically ζ-transitive. □

The converse of Corollary 3.7 is not always true as shown in the following example:

Example 3.10. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then, $\zeta O(X) = \{\phi, X\}$ and $\alpha O(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Consider the constant map $f: X \to X$ defined as $f(x) = c$, for all $x \in X$. Then f is both ζ-irresolute and α-irresolute. Obviously, f is ζ-transitive, but it is not α-transitive because for every positive integer n, we have $f^n(\{a\}) \cap (\{a, b\}) = \phi$.

Theorem 3.11. Let (X, f) be an ζ-dynamical system. Then the following statements are equivalent:

1. f is topologically ζ-transitive,
2. For every nonempty ζ-open set U in X, $\bigcup_{n=1}^{\infty} f^n(U)$ is ζ-dense in X,
3. For every nonempty ζ-open set U in X, $\bigcup_{n=0}^{\infty} f^n(U)$ is ζ-dense in X,
4. For every nonempty ζ-open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is ζ-dense in X,
5. For every nonempty ζ-open set U in X, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is ζ-dense in X,
6. If $E \subseteq X$ is ζ-closed and $f(E) \subseteq E$. then $E = X$ or E is nowhere ζ-dense,
7. If $f^{-1}(U) \subseteq U$ and U is ζ-open in X, then $U = \phi$ or U is ζ-dense in X.

Proof. (1⇒ 2). Assume that $\bigcup_{n=1}^{\infty} f^n(U)$ is not ζ-dense, then by Lemma 2.10, there exists a non-empty ζ-open set V such that $\bigcup_{n=1}^{\infty} f^n(U) \cap (V) = \phi$. This implies that $f^n(U) \cap (V) = \phi$, for every positive integer n, which is a contradiction because f is ζ-transitive. Hence, $\bigcup_{n=1}^{\infty} f^n(U)$ is ζ-dense in X.

(2⇒ 3). Since $\bigcup_{n=1}^{\infty} f^n(U) \subseteq \bigcup_{n=0}^{\infty} f^n(U)$ and by (2), $\bigcup_{n=1}^{\infty} f^n(U)$ is ζ-dense in X. Hence $\bigcup_{n=0}^{\infty} f^n(U)$ is ζ-dense in X.

(3⇒ 1). Let U and V be any two non-empty ζ-open sets in X such that $\bigcup_{n=0}^{\infty} f^n(U)$ is ζ-dense in X. Hence, by Lemma 2.10, we get $\bigcup_{n=0}^{\infty} f^n(U) \cap (V) \neq \phi$. Therefore, f is topologically ζ-transitive.

(3⇒ 4). If U is any non-empty ζ-open set in X and since f is ζ-irresolute, then $f^{-1}(U)$ is an ζ-open in X. Therefore, by (3), we get

$\bigcup_{n=0}^{\infty}(f^{-1})^n(U) \subseteq \bigcup_{n=0}^{\infty} f^{-n}(U)$ is ζ-dense.

(4⇒ 5). Obvious.

(5⇒ 1). Let A and B be any two non-empty ζ-open subsets in X. Since f is ζ-irresolute, then $f^{-1}(A)$ and $f^{-1}(B)$ are ζ-open. Taking $U = f^{-1}(A)$ and $V = f^{-1}(B)$. Then by (5), $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is ζ-dense, this implies that $\bigcup_{n=1}^{\infty} f^{-n}(U) \cap B \neq \phi$. Therefore, $\bigcup_{n=1}^{\infty} f^{n}(A) \cap B \neq \phi$. Whence, f is ζ-transitive.

(1⇒ 6). Let f be ζ-transitive and $E \subseteq X$ be an ζ-closed set such that $f(E) \subseteq E$. Assume that $E \neq X$ and E has a non-empty ζ-interior. If we define $V = X \setminus E$, then V is ζ-open because it is the compliment of an ζ-closed set. Since E has a non-empty ζ-interior, then there exists an ζ-open set W such that $W \subseteq E$. Since E is f-invariant, then we have $f^n(W) \subseteq E$, for every positive integer n. Since f is ζ-transitive, then there is a positive integer n such that $f^n(W) \cap V \neq \phi$. Which is contradiction. Hence, E has empty ζ-interior, implies that E is nowhere ζ-dense.

(6⇒ 7). Let U be a non-empty ζ-open set in X and $f^{-1}(U) \subseteq U$. Suppose that $U \neq \phi$, then $X \setminus U \neq X$. Therefore, we have $X \setminus U$ is ζ-closed and $X \setminus f^{-1}(U) \supseteq X \setminus U$. Hence, $f^{-1}(X \setminus U) = X \setminus f^{-1}(U) \supseteq X \setminus U$ and then $f(X \setminus U) \subseteq X \setminus U$. Hence, by (6), $X \setminus U$ is nowhere ζ-dense implies that U is ζ-dense.

(7⇒1). Suppose that f is not topologically ζ-transitive, then by (3), there exists a non-empty ζ-open set W such that $\bigcup_{n=0}^{\infty} f^{-n}(W)$ is not ζ-dense in X. Taking $U = \bigcup_{n=0}^{\infty} f^{-n}(W)$ and since f is ζ-irresolute, so U is a non-empty ζ-open set, which is not ζ-dense. Hence, we get $f^{-1}(\mathrm{U}) = f^{-1}(\bigcup_{n=0}^{\infty} f^{-n}(W)) = \bigcup_{n=1}^{\infty} f^{-n}(W) \subseteq U$. This contradicts (7). Hence, f is topologically ζ-transitive. □

Definition 3.12. Two ζ-dynamical systems (X,f) and (Y,g) are called ζ-conjugate if there exists an ζ-homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. If h is ζ-irresolute surjection, then (X,f) and (Y,g) are called ζ-semiconjugate.

Example 3.13. Let $X = \{x,y,z\}$, $Y = \{a,b,c\}$, $\sigma = \{\phi,\{y\},\{x,z\},X\}$ and $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{b,c\},X\}$. Define $f : X \to X$ such that $f(x) = f(y) = z, f(z) = y$ and $g : Y \to Y$ such that $g(a) = g(b) = c$ and $g(c) = b$. Then, (X,f) and (Y,g) are two ζ-dynamical systems. Suppose that the map $h : X \to Y$ defined as $h(x) = a, h(y) = b$ and $h(z) = c$, then it can be easily check that h is an ζ-homeomorphism such that $h \circ f = g \circ h$. Hence (X,f) and (Y,g) are conjugates. Moreover, both f and g are topologically ζ-transitive.

Lemma 3.14. Let $f : X \to X, g : Y \to Y$ and $h : X \to Y$, be any three maps such that $h \circ f = g \circ h$, then $h \circ f^n = g^n \circ h$, for each $n \in Z$.

Proof. The proof can be done by induction. Obviously, the equality is true for $n = 1$. Now suppose that $h \circ f^{n-1} = g^{n-1} \circ h$, then we have $h \circ f^n = (h \circ f^{n-1}) \circ f = (g^{n-1} \circ h) \circ f = g^{n-1} \circ (h \circ f) = (g^{n-1} \circ g) \circ h = g^n \circ h$. Hence, the proof. □

Theorem 3.15. Let (X,f) and (Y,g) be two ζ-semiconjugate ζ-dynamical systems. If the map f is topologically ζ-transitive, then g is topologically ζ-transitive.

Proof. Let $h : X \to Y$ be ζ-irresolute surjection. Suppose that f is topologically ζ-transitive and let G, H be any two non-empty ζ-open sets in Y, then $h^{-1}(G)$ and $h^{-1}(H)$ are non-empty ζ-open sets in X. Hence, by Theorem 3.4, there is a positive integer n such that $f^{-n}\big(h^{-1}(G)\big) \cap h^{-1}(H) \neq \emptyset$. Thus, $(h \circ f^n)^{-1}(G) \cap h^{-1}(H) \neq \emptyset$. Therefore, by Lemma 3.14, we get $(g^n \circ h)^{-1}(G) \cap h^{-1}(H) \neq \emptyset$. Then, $h^{-1}\big(g^{-n}(G)\big) \cap h^{-1}(H) \neq \emptyset$. Since h is a surjection, so $g^{-n}(G) \cap H \neq \phi$. Hence, by Theorem 3.4, g is topologically ζ-transitive. □

Theorem 3.16. Let (X,f) and (Y,g) be two ζ-dynamical systems. If $h : X \to Y$ is ζ-open injection, then ζ-transitivity of g implies ζ-transitivity of f.

Proof. Let $h : X \to Y$ be ζ-open injection. Suppose that g is topologically ζ-transitive and let G, H be any two non-empty ζ-open sets in X. Since h is ζ-open injection, then $h(G)$ and $h(H)$ are non-empty ζ-open sets in Y. Hence, by Theorem 3.4, there is a positive integer n such that $g^n\big(h(G)\big) \cap h(H) \neq \emptyset$. Then, by Lemma 3.14, $(h \circ f^n)(G) \cap h(H) \neq \emptyset$. Therefore, $h(f^n(G) \cap h(H) \neq \emptyset$. Since h is injection, then

$f^n\ (G) \cap H \neq \phi$. Hence, by Theorem 3.4, f is topologically ζ- transitive. □

Corollary 3.17. Let (X, f) and (Y, g) be two ζ-conjugate ζ-dynamical systems. Then, the map f is topologically ζ-transitive if and only if g is topologically ζ-transitive.

Proof. Let $h : X \to Y$ be an ζ-homeomorphism. Then, $h^{-1} : Y \to X$ is ζ-irresolute surjection and $h : X \to Y$ is ζ-open injection. Therefore, by Theorem 3.15 and Theorem 3.16, f is topologically ζ-transitive if and only if g is topologically ζ-transitive. □

Definition 3.18. Let (X, f) be an ζ-dynamical system. If there exists $x \in X$ such that the set $\{f^n\ (x) : n \in N\}$ is ζ-dense in X, then x is said to have an ζ-dense orbit and f has an ζ-dense orbit.

Definition 3.19. Let (X, τ) be a topological space and $f : X \to X$ be a function. A point $x_0 \in X$ is said to be a ζ-transitive point of f if the orbit $O(x_0, f)$ is ζ-dense in X.

Example 3.20. Suppose that $X = \{a, b\}, \tau = \{\phi, X, \{a\}, \{b\}\}$ and $\zeta O = \{\phi, X, \{a\}, \{b\}\}$. Define $f : X \to X$ as, $f(a) = f(b) = b$. The orbit of a is $O(a, f) = \{a, f(a), f^2\ (a), \ldots\} = \{a, b\} = X$, hence $\zeta cl(O(a, f)) = X$, so a is an ζ-transitive point.

Proposition 3.21. Let (X, f) be an ζ-dynamical system. If there exists a point $x \in X$ such that the orbit of the point $f(x)$ is ζ-dense in X, then the map f is topologically ζ-transitive.

Proof. Let $x \in X$ and $\zeta cl(O(f(x), f)) = X$. Let U, V be any two non-empty ζ-open subsets of X. Hence, by Lemma 2.10, we have $(O(f(x), f) \cap U \neq \phi$, so there exists a positive integer n such that $f^n\ (x) \in U$. Also, there exists a positive integer m such that $f^m\ (x) \in V$. Suppose that $m \geq n$ so, $x \in f^{-n}\ (U)$, and so $f^m\ (x) \in f^{m-n}(U)$. Therefore, we obtain that $\{f^m\ (x)\} \subseteq f^{m-n}(U) \cap V$ implies that there is a positive integer $(m - n)$ such that $f^{m-n}(U) \cap V \neq \phi$. Similar statements are true when $n \geq m$. This shows that f is topologically ζ-transitive. □

Definition 3.22. A map $f : X \to X$ is called:

1. orbit ζ-transitive if there exists $x \in X$ such that $\zeta cl\big(O(x, f)\big) = X$.
2. strictly orbit ζ-transitive if there exists $x \in X$ such that $\zeta cl\big(O(f(x), f)\big) = X$.

Remark 3.23. Let $f: X \to X$ be any map, then the following statements are true:

1. if f is strictly orbit ζ-transitive, then f is orbit ζ-transitive.
2. f is orbit ζ-transitive if and only if f has an ζ-transitive point.
3. f is strictly orbit ζ-transitive if and only if there exists a point $x \in X$ such that $f(x)$ is an ζ-transitive point of f.

Proof. (1) Suppose that f is strictly orbit ζ-transitive, then there exists $x \in X$ such that $\zeta cl\big(O(f(x), f)\big) = X$. Since $O(f(x), f) \subseteq O(x, f)$, so, $\zeta clO(x, f) = X$. Hence, f is orbit ζ-transitive.

(2) Suppose that f is orbit ζ-transitive, then there exists $x \in X$ such that $\zeta clO(x, f) = X$, and hence, f has ζ-transitive point.

Conversely, suppose that f has an ζ-transitive point, thus there exists $x_0 \in X$ such that $\zeta clO(x_0, f) = X$. Hence, f is orbit ζ-transitive.

(3) Obvious. □

The converse of the Remark 3.23 is not true as shown in the following example:

Example 3.24. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $\zeta O(X) = \{\phi, \{a\}, \{b, c\}, X\}$. Define the map $f: X \to X$ as follows: $f(a) = f(b) = c, f(c) = b$. It is obvious that $O(a, f) = \{a, c\}$, hence $\zeta cl(O(a, f)) = X$, but we have $O(\{f(a)\}, f) = \{b, c\}$ and $\zeta cl(O(f(a), f) = \{b, c\}$. Therefore, f is orbit ζ-transitive, but it is not strictly orbit ζ-transitive.

4. ζ-ISOLATED POINTS AND ζ-TRANSITIVITY

In this section, we introduce the concept of ζ-isolated and quasi-ζ-isolated points in topological spaces. Some properties linked with these concepts and ζ-transitivity are explored.

Definition 4.1. A point $x \in A$ is said to be an ζ-isolated point of a subset A, if there is an ζ-open set U of X such that $x \in U$ and $A \cap U = \{x\}$.

Definition 4.2. In a topological space X, a point $x \in X$ is said to be a quasi-ζ-isolated point of X, if there is an ζ-dense subset A of X such that $x \in A$ and x is ζ-isolated point in A.

Proposition 4.3. A point x in a space X is quasi-ζ-isolated if and only if $\zeta int\zeta cl(\{x\}) \neq \phi$.

Proof. Let x be a quasi-ζ-isolated point of X, then there exists an ζ-dense subset A of X and an ζ-open neighborhood U of X such that $A \cap U = \{x\}$. Observing that $\zeta cl(\{x\}) \cup \zeta cl(A\setminus\{x\}) = \zeta cl(A) = X$ and $\zeta cl(A\setminus\{x\}) = \zeta cl(A\setminus U) \subseteq X\setminus U$. Therefore, $U \subseteq \zeta cl\{x\}$ and hence $\zeta int\zeta cl(\{x\}) \neq \phi$.

Conversely, let $U = \zeta int\zeta cl(\{x\})$. Since $U \neq \phi$, then $x \in U$. Let $A = \{x\} \cup (X \setminus U)$. Since $A \cap U = \{x\}$, then x is ζ-isolated point of A. Also, $\zeta cl(A) = \left(\zeta cl(\{x\}) \cup \zeta cl(X\setminus U)\right) \supseteq \left(U \cup (X\setminus U)\right) = X$. Hence, A is ζ-dense subset of X. Therfore, x is quasi-ζ-isolated point of X. □

Proposition 4.4. A point x in a ζT_1-space X is quasi-ζ-isolated if and only if $\{x\}$ is open and closed.

Proof. Since X in a ζT_1-space, so $\{x\}$ is ζ-closed for all $x \in X$. If x is quasi-ζ-isolated, then by Proposition 4.3, $\zeta int\zeta cl(x) \neq \phi$ implies $\zeta int(\{x\}) \neq \phi$. Hence, $\{x\}$ is both open and semi-closed implies that $\{x\}$ is open and closed.

Conversely, if $\{x\}$ is both open and closed in X, then $\{x\}$ is ζ-open set containing the point x. Hence, we have X is ζ-dense and $X \cap \{x\} = \{x\}$. Therefore, x is quasi-ζ-isolated. □

From Proposition 4.3 and Proposition 4.4, we get the following results:

Corollary 4.5. Let (X,τ) be a ζT_1-space X. Then, every point of X is a quasi-ζ-isolated if and only if (X,τ) is a discrete space.

Corollary 4.6. In an ζT_1-space X, a point x is quasi-ζ-isolated if and only if it is isolated.

Proposition 4.7. Let (X,f) be any ζ-dynamical system in which the face space contains no quasi- ζ-isolated points. Then, a point $x \in X$ is ζ-transitive of f if and only if $f^n(x)$ is ζ-transitive of f for some positive integer number n.

Proof. If $f^n(x)$ is a ζ-transitive point of f, for any given n, then $O(f^n(x),f)$ is ζ-dense. Since $O(f^n(x),f) \subseteq O(x.f)$ for any given n, hence $O(x,f)$ is ζ-dense in X. Therefore, x is an ζ-transitive point of f.

Conversely, assume that x is a ζ-transitive point of f, so $O(x,f)$ is ζ-dense subset of X containing x. Since X contains no quasi-ζ-isolated points, so by Definition 4.2, $O(x,f) \cap U \neq \{x\}$ for every ζ-open set U containing x. Thus, there exists $n \in N$ such that $f^n(x) \in O(x,f) \cap U \subseteq U$. Therefore, $f^n(x)$ is also an ζ-transitive point of f. □

Proposition 4.8. Suppose that X is a topological space having no quasi-ζ-isolated points. Then every orbit-ζ-transitive map f is topologically ζ-transitive.

Proof. Suppose that x is a ζ-transitive point of f, then $O(x,f)$ is ζ-dense subset of X containing x. Let U and V be any two non-empty ζ-open sets in X. Since X contains no quasi-ζ-isolated points, so by Definition 4.2, $O(x,f) \cap U \neq \{x\}$ and $O(x,f) \cap U \neq \{x\}$. Hence, there exist $n,m \in N$ such that $f^n(x) \in U$ and $f^m(x) \in V$. If $m \geq n$, then we have $x \in f^{-n}(U)$ and hence, $f^m(x) \in f^m(f^{-n}(U)) \subseteq V$. Therefore, $f^{m-n}(U) \cap V \neq \phi$. If $n \geq m$, then we obtain that $f^{n-m}(V) \cap U \neq \phi$. Thus, f is an ζ-transitive. □

Proposition 4.9. Let X be a topological space without quasi-ζ-isolated points, and $f: X \to X$. Then the following statements are equivalent:

1) f is orbit-ζ-transitive;
2) f is strictly orbit-ζ-transitive;
3) the set of ζ-transitive points of f is ζ-dense in X.

Proof. (1⇒ 2). If f is orbit-ζ-transitive, then there is $x \in X$ such that $\zeta clO(x,f) = X$. By Proposition 4.8, $f(x)$ is also an ζ-transitive point of f. Hence, f is strictly orbit-ζ-transitive.

(2⇒ 3). Suppose that f is strictly orbit-ζ-transitive and let $A \subseteq X$ be the set of all ζ-transitive points of f. If A is not ζ-dense in X, then there exists an ζ-open set U such that $A \cap U = \phi$. Now, for every $x \in A$, x is ζ-transitive point of f. Hence by Proposition 4.8, for any given $n \in Z^+$, we have $f^n(x)$ is also ζ-transitive point of f, implies that $f^n(x) \in A$, for any $n \in Z^+$. Since $A \cap U = \phi$, so $O(f(x),f)$ is not ζ-dense, which contradicts (2). Therefore, A is ζ-dense in X.

(3⇒1). If f is not orbit-ζ-transitive, this implies that the set of ζ-transitive points of f is empty, which is contradiction because the empty set is not ζ-dense in X. Hence, f is orbit-ζ-transitive. □

From Corollary 4.6 and Proposition 4.9, we get the following result:

Corollary 4.10. Let X be an ζT_1 topological space without isolated points and $f: X \to X$. Then f is orbit-ζ-transitive if and only if f is strictly orbit-ζ-transitive.

Proposition 4.11. Let X be an ζ-regular and let $f: X \to X$ be an ζ-irresolute map. If every proper ζ-closed invariant subset of X is nowhere ζ-dense, then $f(X)$ is ζ-dense.

Proof. Suppose $f(X)$ is not ζ-dense, so there exists $x \in X \setminus \zeta cl(f(X))$ and ζ-regularity gives disjoint ζ-open sets U and V such that $x \in U$ and $(f(X)) \subseteq V$. Taking $W = X \setminus U$, then W is an ζ-closed set and we have, $f(W) \subseteq f(X) \subseteq \zeta cl(f(X)) \subseteq W$, so W is invariant, since $V \subseteq W$, so $\zeta int \zeta cl(W) \supseteq V \neq \phi$ implies W is not nowhere ζ-dense. Hence, W ζ-closed invariant subset of X, which is not nowhere ζ-dense leads to a contradiction. Hence, $f(X)$ is ζ-dense. □

Definition 4.12. Let $f : X \to X$ be any map. A point $x \in X$ is called ζ-non-wondering if for every ζ-open set $U(x)$ containing x, there exists $n \in N$ such that $f^n(U(x)) \cap U(x) \neq \phi$. The set of all ζ-non-wondering points of f is denotes by $\Lambda(f)$.

Example 4.13. In Example 3.22, it is clear that $\Lambda(f) = \{b, c\}$.

Proposition 4.14. Let $f: X \to X$ be a ζ-irresolute map. Then f is topologically ζ-transitive if and only if $\Lambda(f) = X$, and has a ζ-transitive point.

Proof. Let f be topologically ζ-transitive, clearly f has ζ-dense orbit, that is, there exists $y_0 \in X$ such that $O(y_0, f)$ is ζ-dense in X. If $\Lambda(f) \neq X$, then there exists a non-empty ζ-open set V such that $\{f^m(V) : m > 0\}$ are pairwise disjoint sets. Since $O(y_0, f)$ is an ζ-dense orbit, so there is some $k \geq 0$ such that $f^k(y_0) \in V$. Hence, we get $f^{m+k}(y_0) \in f^m(V)$, for m ≥ 0 which is contradiction because $\{f^m(V): m > 0\}$ are pairwise disjoint sets. Therefore, $\Lambda(f) = X$.

Conversely, let f be ζ-dense orbit and $\Lambda_N(f) = X$ and suppose U and V are ζ-open sets in X. Let $x \in X$ be an ζ-dense orbit. Since $O(x, f)$ is ζ-dense, so $O(x, f) \cap U \neq \phi$ and $O(x, f) \cap V \neq \phi$. Assume that k and r are the least integer numbers such that $f^k(y) \in U$ and $f^r(y) \in V$. Let $k < r$, then $m = r - k \in Z^+$ and obviously $f^m(U) \cap V \neq \phi$. Therefore, f is topologically ζ-transitive. □

Definition 4.15. A dynamical system (X, f) is said to be a strongly ζ-transitive if for every non-empty ζ-open set $U \subseteq X$, $\bigcup_{n=1}^{\infty} f^n(U) = X$.

It is obvious that every strongly transitive map is strongly ζ-transitive, but not conversely as it is shown in the following example:

Example 4.16. Suppose that $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}, \{c\}, \{c, d\}\}$.
Then, $\zeta O(X) = \{\phi, X, \{a, b\}, \{c, d\}\}$. Therefore, the map $f : X \to X$ defined as: $f(a) = d, f(b) = c, f(c) = b, f(d) = a$ is strongly ζ-transitive, which is not strongly-transitive $\bigcup_{n=1}^{\infty} f^n(U) \neq X$.

lemma 4.17. If $f: X \to X$ is a strongly ζ-transitive map, then it is topologically ζ-transitive.

Proof. Let U, V be any two non-empty ζ-open subsets of X. Since f is a strongly ζ-transitive map, then $\bigcup_{n=1}^{\infty} f^n(U) \cap V \neq \phi$. Hence, there is a positive integer n such that $f^n(U) \cap V \neq \phi$. Therefore, f is topologically ζ-transitive. □

The converse of Lemma 4.17 is not always true as shown in the following example:

Example 4.18. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}, \{a\}\}$. The family of all ζ-open sets is $\zeta O(X) = \{\phi, X, \{b\}, \{a, c\}\}$. Define the map $f : X \dashrightarrow X$ as follows: $f(a) = f(c) = b, f(b) = c$. f is an ζ-transitive, but not strongly ζ-transitive map by Definition 4.15.

5. CONCLUSION

In this work, we introduced the concept of topological ζ-transitivity by applying the notion of ζ-open sets, which was introduced in [4]. Some properties of such concepts are introduced. also, we introduced the concept of maps with ζ-transitive points, orbit ζ-transitive maps and strictly orbit ζ-transitive maps and we proved that orbit ζ-transitivity is equivalent to ζ-transitivity of maps and both of them are equivalent to the condition that the set of ζ-transitive points of the map is ζ-dense. Moreover, topologically transitive and topologically α-transitive maps are compared with topologically ζ-transitive maps.

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