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SOME RESULTS ON THE EXISTENCE OF RESIDUAL MEASURES

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ABSTRACT

Some results on the existence of residual measures have been generalized. Further results and a characterization of such measures are obtained.

KEYWORDS residual measures, normal measures, hyperdiffuse measures, and category measures.

1. INTRODUCTION AND PRELIMINARIES

In mathematics, generally, the concept of smallness exists and plays an important role. The aim of the present study is to focus on this concept in the sense of topological space and measure theory. Nowhere dense (meager) sets in topological spaces are considered to be small as they contain lots of holes. Null sets in measure theory are small, which are sets of measure zero. Now here a question arises, how these two notions are related? This is why we study residual measures. In such a type of measures, nowhere dense and null sets are similar. Armstrong and Prikry [2] defined residual measures as a generalization to both normal measures of Dixmier [3] and category measures of Oxtoby [9]. Residual measures are also called hyperdiffuse measures, see [4]. After them many other authors have studied this class of measures including [4], [5], [7], [11] and [10]. In this study, we present some results on the existence of residual measures and obtain a characterization of such measures.

Let (X,τ) be a given topological space and let $A \subseteq X$. The notations $Int(A)$ and $Cl(A)$ are, respectively, the interior and the closure of A. A set A is nowhere dense if $Int(Cl(A)) = \emptyset$. Meager set is a countable union of nowhere dense sets. A is regular closed if $A = Cl(Int(A))$. A space that has a countable dense subset is called separable. A point x is isolated of $A \subseteq X$ if there is an open set U in X such that $A \cap U = \{x\}$. X is perfect if it contains no isolated points. A space X is connected if it cannot be the union of two disjoint open sets. X is locally connected if it has a base of connected sets (as subspaces). X is a T_1-space if every $\{x\}$ is closed.

A Borel measure μ is a σ-additive measure defined on the least σ-algebra $\mathbf{B}(X)$ containing all open subsets of a topological space (X, τ). It is called (1) trivial if $\mu(X) = 0$, otherwise it is nontrivial (2) finite if $\mu(X) < \infty$ (3) nonatomic if $\mu(\{x\}) = 0$, for each $\{x\} \in X$, and (4) strictly positive if $\mu(G) > 0$, for each $G \in \tau$.

A measure μ is regular on X if for every $B \in \mathbf{B}(X)$ we have that

$$\mu(B) = \inf\{\mu(G) : B \subseteq G, G \text{ is open}\}$$
$$= \sup\{\mu(F) : F \subseteq B, F \text{ is closed}\}.$$

A subset A of a measure space $(X, \mathbf{B}, \mu)$ is Jordan measurable if $\mu(\partial(A)) = 0$, where ∂ stands for the topological boundary of a set. A null is a set of measure zero. A function f is continuous almost everywhere if the set of discontinuity of f has measure zero. The support of μ, denoted by supp(μ), is the set of all points $x \in X$ for which every open set U containing x is of positive measure.

Remark 1.1. *The support of μ has the following properties:*

(1) *$supp(\mu)$ is closed because it is the complement of the largest open set of measure zero (largest means union of all open sets of measure zero).*

(2) *Every nonempty open set in $supp(\mu)$ is of positive measure.*

(3) *$supp(\mu)$ has a full measure, i.e.* $\mu(supp(\mu)) = 1$.

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Throughout this study, the word "space" stands for an arbitrary topological space without assuming any type of separation axiom, and all measures are assumed to be finite Borel.

2. RESIDUAL MEASURES

Definition 2.1. *[2] A measure μ on some topological space X is said to be residual if $\mu(N) = 0$ for every nowhere dense subset N of X.*

The following are examples of residual and nonresidual measures.

Remark 2.2. The induced measure on the Stone space of the measure algebra of the Lebesgue measure on the unit interval is a nice example of residual measures, whereas the Lebesgue measure itself is not residual. The so-called fat Cantor set [6, Page 39] is nowhere dense with positive (Lebesgue) measure.

Next, we call back some important results of residual measures from the literature.

Theorem 2.3. *[7] If X is a perfect metric space, then no nontrivial residual measure can be defined on X, (cf. [4, Proposition 5]).*

Theorem 2.4. *[4] If X is a locally connected space, then no nontrivial regular nonatomic residual measure can exist on X.*

Now here a question arises: Can a similar result (to the above one) be true for connected spaces? The question has already been answered by Plebanek in [10] under a set theoretic assumption. Namely, assuming the continuum hypothesis, there is a perfectly normal compact connected space X on which there is a strictly positive residual measure.

Next, we provide another easy example of a connected space that supports a residual measure.

Example 2.5. *Let μ be the Lebesgue measure on the density topological space $(\mathbf{R}, \tau_d)$. Then τ_d is the family of all measurable subsets E of $\mathbf{R}$ such that each $x \in E$ has following property:*

$$\lim_{n\to 0} \frac{\mu(E \cap (x-h, x+h))}{2h} = 1.$$

It is reported in [9, Page 100] that $(\mathbf{R}, \tau_d)$ is connected. By [8, Lemma 4.3.1], every nowhere dense set N in τ_d has measure zero, i.e. $\mu(N) = 0$, which proves that μ is residual on $\mathbf{R}$.

Note that a similar result to the following was proved in [7, Proposition 3] for regular residual measures, we prove it for a more general class of measures.

Theorem 2.6. *Let X be a separable T_1-space. There is a nontrivial residual measure if and only if X has at least one isolated point.*

Proof. Let $x \in X$ be an isolated point. Then the Dirac measure δ_x is a nontrivial residual measure. Conversely, suppose contrary that X has no isolated points, and assume that X is a separable T_1 space. Then X has a countable dense subset D. Set $D = \cup_{i\in\omega} \{x_i\}$. Since X is T_1, every $\{x_i\}$ is closed. Since X has no isolated points, no $\{x_i\}$ can also be open, and so Int$(\{x_i\}) = \emptyset$. This means that all $\{x_i\}$ are nowhere dense sets. Let μ be a nontrivial residual measure on X. Then $\mu(\{x_i\}) = 0$. Now

$$\mu(D) = \sum_{i\in\omega} \mu(\{xi\}) = 0.$$

Since D is dense in X, then $X \setminus D$ is nowhere dense (Borel) (being the complement of a Borel set D which is a countable union of closed sets). Thus $\mu(X \setminus D) = 0$. Therefore $\mu(X) = \mu(X \setminus D) + \mu(D) = 0$. It follows that μ is a trivial measure, which is contradiction. Since μ was taken arbitrarily, the proof follows. □

Theorem 2.7. *Let X be a separable space. There exists no nontrivial nonatomic residual measure on X.*

Proof. Suppose that X is a separable space. Then X has a countable dense subset D. Namely, $D = \{x_1, x_2, x_3, \ldots\}$. Let μ be a nontrivial nonatomic residual measure on X. Then $\mu(\{x_i\}) = 0$, for $i = 1, 2, 3, \ldots$ (by nonatomicity). Now

$$\mu(D) = \sum_{i=1}^{\infty} \mu(\{x_i\}) = 0.$$

This implies that D is Borel measurable (a countable union of closed sets). Since D is dense in X, $X \setminus D$ is nowhere dense Borel (The complement of a Borel set is also Borel). Since μ is residual, so $\mu(X \setminus D) = 0$. Therefore $\mu(X) = \mu(X \setminus D) + \mu(D) = 0$, which contradicts to our assumption that μ is nontrivial nonatomic residual measure. This completes the proof. □

Definition 2.8. *[1] A measure μ on a topological space X is said to be uniformly regular if there is a countable family $\mathcal{G}$ of open subsets of X such that for every open set $U \subseteq X$ and every $\varepsilon > 0$, there is $G \in \mathcal{G}$ with $G \subseteq U$ such that*

$$\mu(U \setminus G) < \varepsilon.$$

Proposition 2.9. *Let μ be a strictly positive uniformly regular measure on a space X. Then X is separable.*

Proof. Let G = $\{G_n : n \in \mathbf{N}\}$ be a family of open subsets of X that makes μ uniformly regular. For every G_n choose an element $x_n \in G_n$. Set $D = \{x_n : n \in \mathbf{N}\}$. Obviously, D is countable. It remains to show that this set is dense in X. Suppose otherwise that $X \neq Cl(D)$. So $X \setminus \mathrm{Cl}(D)$ is a nonempty open subset in X. By assumption, there is a nonempty open set G_n such that $x_n \in G_n \subseteq X \setminus \mathrm{Cl}(D)$, which is contraction. Hence $X = \mathrm{Cl}(D)$. This proves that X is separable.□

Theorem 2.10. *Let (X, τ) be a topological space. Then there is no nontrivial (strictly positive) nonatomic uniformly regular measure μ on X that is also residual.*

Proof. Suppose contrary that there exist a nontrivial nonatomic uniformly regular μ on X. By Proposition 2.9, X is a separable space and by Theorem 2.7, $\mu = 0$, contradiction! The proof follows. □

Proposition 2.11. *On discrete spaces, every measure is residual.*

Proof. Follows from the fact that the only nowhere dense set in a discrete space is $\emptyset$, and $\mu(\emptyset) = 0$ for all measures μ.□

Theorem 2.12. *Let μ be a residual measure on a topological space (X, τ). Then the support of μ (if exists), supp(μ), is a regular closed set in X.*

Proof. Let $F = \mathrm{supp}(\mu)$. Then $\mathrm{Int}(F) \subseteq F$ and F is closed by Remark 1.1 (1). Therefore $\mathrm{Cl}(\mathrm{Int}(F)) \subseteq F$. Consider $F \setminus \mathrm{Cl}(\mathrm{Int}(F))$ is nowhere dense. By assumption $\mu(F \setminus \mathrm{Cl}(\mathrm{Int}(F))) = 0$. By Remark 1.1 (3), $\mu(X \setminus \mathrm{Cl}(\mathrm{Int}(F))) = 0$. Again by Remark 1.1 (1), the complement of F, $X \setminus F$, is the largest open set, which implies that $X \setminus \mathrm{Cl}(\mathrm{Int}(F)) \subseteq X \setminus F$ and so $F \subseteq \mathrm{Cl}(\mathrm{Int}(F))$. Hence $F = \mathrm{Cl}(\mathrm{Int}(F))$. Thus F is regular closed.□

3. A CHARACTERIZATION OF RESIDUAL MEASURES

Lemma 3.1. *For any open set G of a topological space X,* $\mathrm{Int}(\mathrm{Cl}(G) \setminus G) = \emptyset$.

Proof. Let G be an open set. To show that $\mathrm{Int}(\mathrm{Cl}(G) \setminus G) = \emptyset$, suppose otherwise that there exists a nonempty open set H such that $H \subseteq \mathrm{Cl}(G) \setminus G$. Then $G \subseteq \mathrm{Cl}(G) \setminus H$ and $\mathrm{Cl}(G) \setminus H$ is closed. Therefore $\mathrm{Cl}(G) \subseteq \mathrm{Cl}(G) \setminus H$ which is contradiction. □

Theorem 3.2. *Let (X, τ) be a topological space and let μ be a regular Borel measure on X. The following are equivalent:*

(1) *Every open subset U of X is Jordan measurable.*

(2) *Every closed subset F of X is Jordan measurable.*

(3) *Every (Borel) measurable subset E of X is Jordan measurable.*

(4) *Every real-valued measurable function f is continuous over an open set of full measure.*

(5) *Every real-valued measurable function f is continuous almost everywhere.*

(6) *Every nowhere dense (Borel) subset N of X is null. That is μ is residual.*

(7) *The closure of every nowhere dense (Borel) subset N of X is null.*

(8) *Every meager (Borel) subset M of X is null.*

Proof. (1) ⇔ (2) Let F be a closed set. Then $\mu(\partial(F)) = \mu(\partial(X \setminus F)) = 0$. Set $U = X \setminus F$. Thus U is open and Jordan measurable.

(2) ⇒ (3) Let E be a measurable set. Since μ is regular, for every $e > 0$, there is a closed set F with $F \subseteq E$ such that $\mu(E \setminus F) < \epsilon$. By (2), $\mu(F) = \mu(\mathrm{Int}(F))$, so $\mu(E \setminus \mathrm{Int}(F)) = \mu(E \setminus F) < \epsilon$. Hence $\mu(E \setminus \mathrm{Int}(E)) < \epsilon$. Since ϵ was selected arbitrarily, let $\epsilon \to 0$. Therefore,

$$\mu(E) = \mu(\mathrm{Int}(E)) \;.\;.\;.\;.\;.\;.\; (*)$$

By the same way above using the complement of (2), we can get

$$\mu(E) = \mu(\mathrm{Cl}(E)) \;.\;.\;.\;.\;.\;.\; (**)$$

It follows from (∗) & (∗∗) that $\mu(\mathrm{Cl}(E)) = \mu(\mathrm{Int}(E))$. This implies that $\mu(\partial(E)) = 0$. Hence E is Jordan measurable.

(3) ⇒ (4) Let $f : X \to \boldsymbol{R}$ be a measurable function. For $p, q \in \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers, set $E_{p,q} = \partial(f^{-1}(p, q))$, $F = \bigcup_{p,q \in \mathbf{Q}} E_{p,q}$ and $D = \mathrm{Cl}(F)$. Since $f^{-1}(p, q)$ is measurable, by (3) $E_{p,q}$ is of measure zero (because $E_{p,q}$ is Jordan measurable). Again, by (3) $\mu(D) = 0$ for the countable set $\mathbf{Q}$. Let $x \in X \setminus D$ and $p, q \in \mathbf{Q}$ such that $p < f(x) < q$. Eventually, $x \in \mathrm{Int}(f^{-1}(p, q)))$. If not, we would have $x \in \partial(f^{-1}(p, q)) = E_{p,q} \subseteq D$, which is impossible. As $X \setminus D$ is open and $\mu(X \setminus D) = 1$, we have shown that for every f is continuous at every $x \in X \setminus D$. This completes the proof.

(4) ⇒ (5) Suppose that (4) is true. Let G be such open set with $\mu(G) = 1$. Set $D = X \setminus G$. Then $\mu(D) = 0$. By (4) f is continuous at every x which is not in D. This is exactly the definition of continuity almost everywhere. We are done.

(5) ⇒ (6) Let N be a Borel measurable subset of X. The indicator function χ_N of N is measurable. By (5) χ_N is continuous almost everywhere. Let $A_0 = \{x : \chi_N(y) = 0, y \in U\}$ and $A_1 = \{x : \chi N(y) = 1, y \in U\}$, where U is a neighborhood. Since {0} and {1} are open (because the topology on range of χ_N is discrete), so A_0 and A_1 are open. Let C be the set of points of continuity of χ_N. We have that $C \subseteq A_0 \cup A_1$. Clearly, $A_1 \subseteq N$ and $A_0 \subseteq X \setminus N$. If N is also a nowhere dense set, then $A_1 = \emptyset$ and so $C \subseteq A_0 \subseteq X \setminus N$. Since $_\mu(C) = 1$, therefore $\mu(X \setminus N) = 1$. But $N \subseteq X \setminus C$ and $_\mu(X \setminus C) = 0$. Thus $\mu(N) = 0$. This shows that μ is residual.
(6) ⇔ (7) This follows from that fact that a set is nowhere dense if and only if its closure is nowhere dense.
(7) ⇔ (8) This equivalence can be found by the definition of meager sets and σ-additivity of μ.

(8) ⇒ (1) Let U be an open subset of X. By Lemma 3.1, $\mathrm{Cl}(U)\setminus U$ is nowhere dense (meager). By (8), $\mu(\mathrm{Cl}(U)\setminus U) = 0$. Hence $\mu(\partial(U)) = 0$, i.e., U is Jordan measurable.☐

We shall remark that a part of the above characterization was proved by Zindulka in [11, Proposition 2.6] for a measure called Jordan that is stronger than the residual measure. We follow his proof and provide additional improvements. It is worth saying that the Jordan measure employed his characterization is not the classical one otherwise it would be false. There is a measurable set that does not satisfy (2), see [1, Page 68].

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