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STABILITY OF RICCATI STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT

In this work, we study the numerical method for solving stability of Riccati stochastic differential equation, because of the difficulty of finding analytical solutions for many of the stochastic differential equations the Riccati method was used. Numerical simulations for different selected examples are implemented. In addition, the absolute error, the linear stability are supported by numerical tests problems.

KEYWORDS: Numerical Solutions, Riccati stochastic differential equations , linear stability.

0. INTRODUCTION

A large class of dynamical systems appearing throughout the field of engineering and applied mathematics is described by the second order differential equation which has the form:

$$y'' + p(x)y' + q(x)y = r(t) \quad \ldots (1)$$

where

p, q, and (r) are real functions on R. In general, there is no method for solving

Nonhomogeneous linear second order differential equations and, therefore, a complete analysis of (1) does not exist. Nevertheless, in the homogeneous case, when r= 0 in (1) by making the change of variable $= {y'}/{y}$, we are led to a first order differential equation of the form

$$y' = -p(t)y + y^2 + q(t) \quad \ldots (2)$$

Although the analysis of these kinds of differential equations are still in a prelim-nary stage, recently various issues concerning theoretical aspects of such differential equations have been successfully clarified. In the literature, (2) is a special case of a more general one, so-called scalar Riccati differential equation, namely

$$y' = a(t)y + b(t)y^2 + c(t) \quad \ldots (3)$$

where a ,b, and c are real functions on R. The study of (3) has long been an important topic and dates from the early period of modern mathematical analysis. It began with examinations of particular cases of (1) by [1] and then by [2,4]. The generalization of scalar Riccati differential equation to the case gives us Riccati differential equation which is one of the central objects of present day control theory. In fact, in the theory of control systems, the qualitative control problem has received considerable research interests. This problem is regarded as an extension of the classical result [1]. [10] on controllability and stability of linear systems which is relevant to such differential equations (see [5, 16, 4, 3, 7, 8]). Riccati differential equations also play predominant roles in other control theory problems such as dynamic games, linear systems with Markovian jumps, and stochastic control. The study of such differential equations, which also appears in a number of other areas such as biomathematics and multidimensional transport processes, is an interesting area of current research. There exists a rather extensive literature on the Riccati differential equation, mainly developed within the automatic control literature. We refer the readers to [3] as an extensive survey as well as to [5, 16, 4,9]as fundamental papers on this area.

Our work ,we study the Stability analysis of stochastic Riccati differential equation, which has the form:-

$$y' = p(x)y^2 + (q(x) + r(x)w_t)y + h(x)\,;\, x \in [0, T] \quad \ldots (4)$$

where the functions p(x),q(x) and h(x) are continuous in x . Moreover r(x) is a function which is defined on the same interval and w_t is a white noise process.

2.<u>Main Results.</u>

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In this section we can state and prove a theorem by using numerical stochastic differential equation.

Theorem1.

Suppose that $y' = p(x)y^2 + (q(x) + r(x)w_t)y + h(x)\,; x \in [0, T]$. where the functions p(x),q(x) and h(x) are continuous in x. And w_t is a white noise process, then

$$Y_{n+1} = y(\tau_n) + [(-2p(\tau_n)u(\tau_n) - q(\tau_n))z_\tau - p(\tau_n)](\tau_{n+1} - \tau_n) - r(\tau_n)z_\tau(w_{t_{n+1}} - w_{t_n})$$

For n=0,1,2,…….,N-1; with initial value $Y_0 = X_0$.

Proof.

Assume that $y = u(x)$, of the Riccati equation(4) is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new dependent variable z.

suppose that $y = u(x)$ solves the RSDE's

let $z = \frac{1}{y-u}$

Then $y = u(x) + \frac{1}{z}$ and $y' = u'(x) - \frac{z'}{z^2}$ substituting equation (4) we get:

$$u'(x) - \frac{z'}{z^2} = p(x)\left[u(x) + \frac{1}{z}\right]^2 + (q(x) + r(x)x_t)\left[u(x) + \frac{1}{z}\right] + R_1(x)$$

$$u'(x) - \frac{z'}{z^2} = p(x)u(x)^2 + 2p(x)u(x)\frac{1}{z} + p(x)\frac{1}{z^2} + q(x)u(x) + q(x)\frac{1}{z} + r(x)x_t u(x) + r(x)w_t\frac{1}{z} + r_1(x)$$

$$-\frac{z'}{z^2} = [p(x)u(x)^2 + q(x)u(x) + r(x)x_t u(x) + r_1(x) - u'(x)] + [2p(x)u(x) + r(x) + r(x)w_t]\frac{1}{z} + p(x)\frac{1}{z^2}$$

Where $[p(x)u(x)^2 + q(x)u(x) + r(x)x_t u(x) + r_1(x) - u'(x)] = 0$

since $u(x)$solves the original SDE ,then $u(x) = p(x)u^2 + (q(x) + r(x)w_t)u + r_1(x)$

Hence ,

$$-\frac{z'}{z^2} = [2p(x)u(x) + r(x) + r(x)w_t]\frac{1}{z} + p(x)\frac{1}{z^2} \qquad \ldots.(5)$$

From equation(3) we get:

$$z' = -[2p(x)u(x) + q(x) + r(x)w_t]z - p(x)$$

since $z' = dz_t$; Then

$$dz_t = ([-2p(x)u(x) - q(x)]z - p(x))dt - r(x)zdw_t \qquad \ldots(6)$$

So , equation (6) take integral both sides we get:

$$z_t = z_{t_0} + \int_{t_0}^{t}([-2p(s)u(s) - q(s)]z_s - p(s))\,ds - \int_{t_0}^{t} r(s)\,z_s\,dw_s$$

Where initial value $z_{t_0} = z_0$ subdivide the interval $[t_0, T]$ into N-subintervals according to the following discretization

$t_0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots < \tau_N = T$

The Riccati approximation is defined as a continuous time stochastic process $Y = \{ Y(T), t_0 \le t \le T\}$satisfying the iterative scheme :

$$Y_{n+1} = y(\tau_n) + [(-2p(\tau_n)u(\tau_n) - q(\tau_n))z_\tau - p(\tau_n)](\tau_{n+1} - \tau_n) - r(\tau_n)z_\tau(w_{t_{n+1}} - w_{t_n})$$

For n=0,1,2,…….,N-1; with initial value $Y_0 = X_0$.

For example. consider

$$\left.\begin{array}{l} dz = ((-2p(t)\,u(t) - q(t))z - p(t))dt - R(t)\,z\,dw \\ z(0) = 0 \end{array}\right\}$$

Then the unique solution is:

$x(t) = e^{\int_0^t(-2p(t)u(t)-q(t)+R^2(t))ds + \int_0^t R(t)dw}$, for $0 \le t \le T$.

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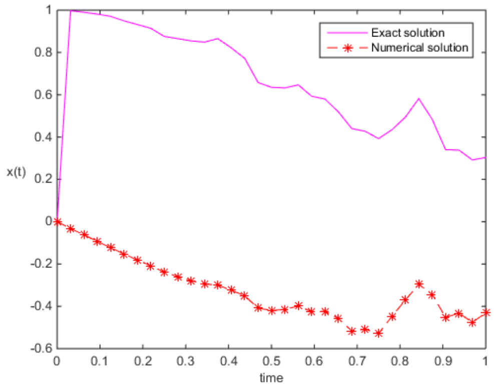
where $p(t) = \cos t$; $u(t) = \sin t$; $q(t) = t$; $R(t) = t^2$; $z_0 = 0$; $Y_0 = 0.$; As discussed previously in illustration (I), the following table (A) is needed for error analysis and as follows

Table (A): Error generated by the Riccati scheme

R	N	Error at final time
1	2^5	0.7351
	2^6	1.0775
	2^8	0.8157
	2^9	0.9187
	2^{10}	0.8252
	2^{11}	0.8904



Fig(1): Exact solution and the numerical solution by Riccati scheme with $N = 2^5$; $R = 1$

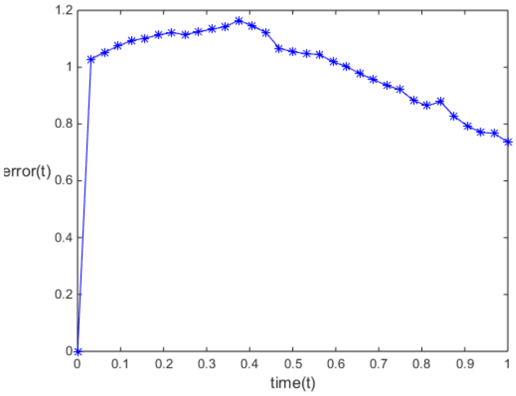


Fig (1.1): Absolute error between the Riccati scheme and exact solution with N = 25; R = 1.

Another example of Illustration. (With Absolute Error Test and Comparisons):

That is:-

$$\left.\begin{array}{l} dz = (-q(t))\, z)dt - R(t)\, z\, dw \\ z(0) = 0 \end{array}\right\}$$

The unique solution is:

$$x(t) = e^{\int_0^t(-q(t)+R^2(t))ds + \int_0^t R(t)dw}, \text{ for } 0 \le t \le T.$$

where: $R(t) = \sin t$; $q(t) = \cos t$; $z_0 = 1$; $Y_0 = 0$; As discussed previously in illustration (II), the following table (B) is needed for error analysis and as follows:

Table (B): Error generated by the Riccati scheme.

R	N	Error at final time

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1	2^4	0.4161
	2^5	0.1964

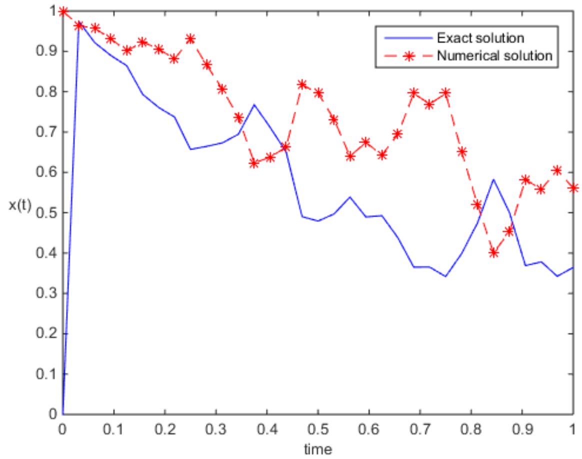

Fig (2): Exact solution and the numerical solution by Riccati scheme with N = 25; R = 1.

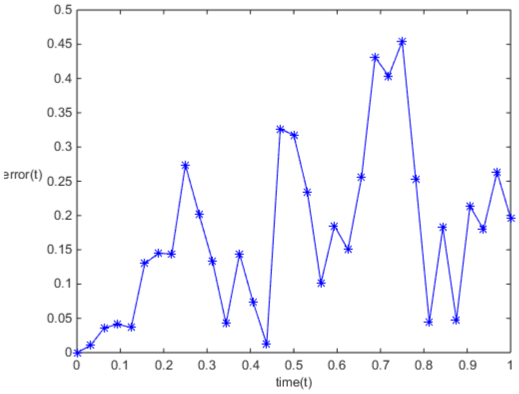

Fig (2.1): Absolute error between the Riccati scheme and exact solution with $N = 2^5$; R = 1

3.Numerical Steps of Liner Stability for RM:

1. The concepts of strong and weak convergence concern the accuracy of numerical methods over a finite interval [0, T], for small step sizes Δt. However, in many applications the long term, $t \longrightarrow \infty$, behavior of an SDE is of interest.

2. Convergence bounds of the form:

$E|X_n - X(T)| \le C\Delta t^{\gamma}$ or $|Ep(X_n) - Ep(X(T))| \le C\Delta t^{\gamma}$

are not relevant in this context, since generally, the constant C grows unboundedly with T.

3. We return to the linear SDE:

$dX(t) = ((-2p(t)\,u(t) - q(t))z - p(t))dt + \mu(X(t))\,dW(t),\ X(0) = X_0$

where the function of RSDE allowed to be complex in the case where $\mu = 0$ and X_0 is constant, (7) reduces to the deterministic linear test equation, if we use the term stable to mean that $\lim_{t\to\infty} X(t) = 0$, for any X_0. Then we see that stability is characterized by $R\{\lambda\} < 0$.

4. We will consider the two most common measures of stability; Mean-Square and Asymptotic,[7]. Assuming that $X_0 \neq 0$ with probability 1, solutions of SIDE is:

$dX(t) = ((-2p(t)\,u(t) - q(t))z - p(t))\,dt + \mu X(t)\,dW(t),\ X(0) = X_0$

Satisfying:

$$\lim_{t\to\infty} EX^2(t) = 0 \Leftrightarrow R\{\int_{t_0}^{t} \lambda\} + \frac{1}{2}|\mu|^2 .0 \quad (7)$$

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$\lim_{t\to\infty} |X(t)| = 0$ with probability 1 $\Leftrightarrow R\{\int_{t_0}^{t} \lambda - \frac{1}{2}\mu^2\} < 0$

The left-hand side of (8) defines what is meant by mean-square stability. The right-hand side of (8) completely characterizes this property in terms of the function SDE. Similarly (9) defines and characterizes asymptotic stability

Consider the following example of Mean-Square stability:

$$dx = (-2p(t)u(t)z)dt + g\,x\,dw$$

$$x(0) = 1$$

where $g(t)=\sin t$; $T = 20$; $u(t) = \mu = 50000$; $X0= 1$; $\Delta t = 1, 2, 1/4$; and $N = T/\Delta t$; $0 \le t \le T$. …(9)

and Asymptotic stability:

where $p(t) = \cos t$; $u(t) = g(t) = \sin t$; $T = 500$; $\Delta t = 1, 2, 1/4$; and $N = T / \Delta t$

The Figure (5) plots the sample average of $E(X^2)$ against t in this picture the $\Delta t = 1$ and $\Delta t = 1/2$ curves increase with t, while the $\Delta t = 1/4$ curve decays toward zero

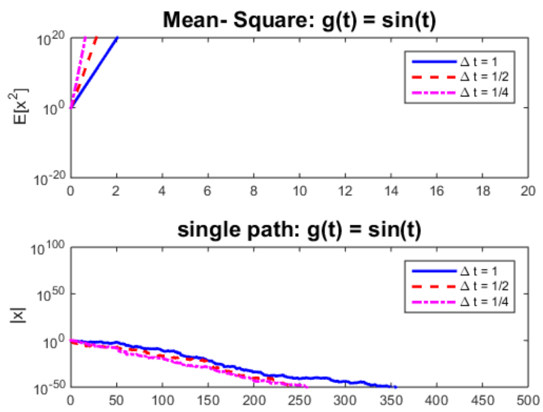

***Fig (5)** Mean-Square and Asymptotic stability.*

CONSULTATION

Numerical methods for the solution of stochastic certain- differential equations are essential for the analysis of random phenomena. Strong solvers are necessary when exploring characteristics of systems that depend on trajectory-level properties. Several approaches exist for strong solvers, in particular Riccati type methods , although both increase greatly in complication for orders greater than one. In many _ financial applications, major emphasis is placed on the probability distribution of solutions, and in particular mean and variance of the distribution. In such cases, weak solvers may sauce. Independent of the choice of stochastic certain- differential equation solver, methods of variance reduction exist that may increase computational efficiency. The replacement of pseudorandom numbers with quasi random analogues from low-discrepancy sequences is applicable as long as statistical independence along trajectories is maintained. In addition, control variates offer an alternate means of variance reduction and increases inefficiency simulation of stochastic certain- differential equations trajectories.

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الخلاصة

في هذا العمل ، ندرس الطريقة العددية لحل ثبات معادلة التفاضلية العشوائية .Riccati. بسبب صعوبة إيجاد حلول تحليلية للعديد من المعادلات التفاضلية العشوائية ، تم استخدام طريقة .Riccati. يتم تنفيذ عمليات المحاكاة العددية لمختلف الأمثلة المختارة .بالإضافة إلى ذلك ، الخطأ المطلق ، الاستقرار الخطي مدعوم بمشاكل الاختبارات العددية.

پوخته

دئەڤی كاریدا , ئەم ڕابووینەب ڤەكۆلینا ڕێكا ژمارەی بۆ چارەسەكرنا هاوكێشێن جیكاكارییێن دیاركری . ژئەگەری زەحمەت دیتنا چارەسەرییێن شیكاری ژبۆ گەلەك ژها وكێشێن جیاكارییێن ڕەمەكی , ڕێكا (ریكاتی)هاتیە بكارهێنان كو دڤێژێدا لاسایكرنا ژمارەی ژبۆ هەمی نموونەرییێن هەلبژارتی هاتیە جێبە جێكرن . زێدەباری ئەڤێ چەندێ كو هاریكارە بۆ چارەسەكرنا چەوبیا نیزێگ بوونا ب هێزئەوا پەیدادكەت