Journal of University of Duhok, Vol. 22, No.1 (Pure and Eng. Sciences), Pp 26-29, 2019
https://doi.org/10.26682/sjuod.2019.22.1.4

ASYMPTOTIC STABILITY AND BOUNDEDNESS OF SOLUTION OF 3rd ORDER NONLINEAR DIFFERENTIAL EQUATION

SIZAR ABID MOHAMMED
Dept. of Mathematics, College of Basic Education, University of Duhok, Kurdistan Region-Iraq

(Received: September 26, 2018; Accepted for Publication: February 10, 2019)

ABSTRACT

In this paper by using the Lyapunov function we investigate the asymptotical stability and boundedness of solution or 3rd order nonlinear differential equation. By using Lyapunov direct method to determine the asymptotic stability and bounded of solution for nonlinear differential equation by using a suitable Lyapunov function.

KEYWORDS: Nonlinear, asymptotical stability, boundedness, Lyapunov function.

INTRODUCTION

In the last years, a lot of interesting results related to the qualitative behaviors of solutions; stability, instability, asymptotically stability, exponentially stability, etc., of nonlinear 3rd order differential equation. As we know the boundedness of solution and stability are very important in the applications of differential equation using Lyapunov direct method. We can determine the stability for nonlinear differential equation without solving it only by define or finding a suitable Lyapunov function.[2,3,6,8,10]

Consider the 3rd order nonlinear differential Eq. (1) with continuous functions $\propto, g$ and Q.

$$u''' + \propto(u,u')u'' + g(u,u') = Q(t;u,u',u'') \qquad (1)$$

In addition the initial condition guaranteed the continuous existence and uniqueness of solution.

In this paper i try to give some simplification to Barbashin theorem [1] also to expand the results in [4] and [5] for discussing the boundedness of solution of Eq. (1) on a real line. In [4] and [5] Eq. (1) is simplified and reduced in order as a system of 1st order by assuming that

$$u' = v$$
$$v' = z$$
$$z' = -\propto(u,v)z - g(u,v) + Q(t;u,u',u'') \qquad (2)$$

For this system, a new Lyapunov function is obtained.

Some basic definition:[3,4,5]

Definition (1): Suppose that a system of first order differential equation $u' = f(t,u)$ defined in R^n and the Lyapunov function w defined as

$$w: I \times R^n \to R \ , \ u \in R^n$$

1) $w(t,u) \geq 0$,
2) $w(t,u) = 0$, if and only if $u = 0$,
3) $w'(t,u) \leq -c|u|$, where $c > 0$, and $|u| = (\sum_{i=1}^{n} u_i^2)^{\frac{1}{2}}$

Definition(2): A Lyapunov function is any continuous and differentiable function $W : G \to R$, such that $W(t,u)$ is positive defined for all $t \in [t_0, +\infty), x \in R^n$, and the time derivative of this function is negative semi -definite or negative definite or positive definite.

Definition(3): The zero solution of Eq. (1) is asymptotic stable if it is stable and if it approaches zero as $t \to \infty$.

For finding the asymptotically stable of Eq. (1) we suppose that $Q(t;u,v,z) \equiv 0$ this means that Eq. (1) is in a homogeneous case.

Theorem (1): Assume that g and $\propto$ are continuous functions, and let $I_0 = [\delta, J]$, $J = \beta k\varepsilon - \beta k\varepsilon^2$, δ, k, ε and β are positive constant. Additionally the following assumptions are hold:

1) $g_u = \frac{g(u,v)-g(0,v)}{u} \in I_0 = \gamma \ , u \neq 0$,
2) $g_v = \frac{g(u,v)-g(u,0)}{v} \in I_0 = \beta \ , v \neq 0$,
3) $g(0,v) = g(u,0) = 0$ and $|\propto(u,v)| \leq k$.

Then the trivial solution of Eq. (1) is asymptotic stable.

In the case of $Q(t;u,v,z) \neq 0$ this means that non homogeneous case.

Theorem (2): Suppose that all assumption of theorem (1) are satisfy and $Q(t;u,v,z) \leq M$, where $M > 0$, then $0 < M < \infty$, and depend only on k ,β and δ, such that all solution of Eq. (1) satisfies

 sizar@uod.ac

$$u^2(t)+v^2(t)+z^2(t) \le e^{-\frac{1}{2}\mu t}\{M_1+M_2\int_{t_0}^{t}|P(\tau)|e^{\frac{1}{2}\mu\tau}d\tau\}^2$$

$t \ge t_0$, $M_1, M_2 > 0$.

Theorem (3): By the assumption of theorem (2) and putting $|Q(t;u,v,z)| = (|u|+|v|+|z|)Q(t)$, where $Q(t)$ is a positive continuous function and satisfy $\int_0^t Q(s)ds \le A < \infty, A > 0$.

Therefore, there exist a constant M_0 such that all solution $x(t)$ of Eq. (1) satisfy $|u(t)| \le M_0$, $|u'(t)| \le M_0$, $|u''(t)| \le M_0$, for sufficient large t.

To prove above theorems, by finding the suitable Lyapunov function $w = w(u,v,z)$. Which i obtained below after some lengthy algebraic computations,[1].

$$2w = \frac{a\delta}{\Delta}\{[\beta^2(1-\varepsilon)^2]u^2 + \{(1-\varepsilon)[k^2-\beta(1-\varepsilon)] + \beta]v^2 + z^2 + 2k\beta(1-\varepsilon)^2uv + 2(1-\varepsilon)^2\beta uz + 2k(1-\varepsilon)vz\}$$

Where a,δ,k,β,Δ and ε are all positive for all u,v,z with $\delta > 1$ and $\Delta = \alpha\beta(\delta-1)(1-\varepsilon)^2$.

Subject to the assumption of theorem (1) there exist positive constant $A_i = A_i(a,\delta,k,\beta,\Delta,\varepsilon)$, $i = 1,2$ such that

$$A_1(u^2+v^2+z^2) \le w(u,v,z) \le A_2(u^2+v^2+z^2)$$

By rearranging Lyponov function we have

$$w = \frac{a\delta}{2\Delta}\{[\beta(1-\varepsilon)u + k(1-\varepsilon)v + z]^2 + \beta^2(1-\varepsilon)^2u^2 + \varepsilon[(1-\varepsilon)k + \beta\varepsilon]v^2 - \varepsilon\beta(1-\varepsilon)uz\}$$

$$w = \frac{a\delta}{2\Delta}\{[\beta(1-\varepsilon)u + k(1-\varepsilon)v + z]^2 + \beta^2\varepsilon(1-\varepsilon)^2u^2 - \beta\varepsilon(1-\varepsilon)(u+\frac{1}{2}z)^2 + \varepsilon[(1-\varepsilon)k + \beta\varepsilon]v^2 + \beta\frac{\varepsilon^2(1-\varepsilon)}{4}z^2\}$$

Which reduces to

$$w \le A_2(u^2+v^2+z^2)$$

Where $A_2 = \frac{\delta}{2\Delta}\max\{\beta^2\varepsilon^2(1+\beta+k),(1-$
εkk+1+β1−εk−11+1−εk+β1−ε.

Lemma(1): Assume that all assumption of theorem (1) holds and A_3 is any positive constant, therefore for any solution of the system (2)

$w' = \frac{d}{dt}w(u,v,z) \le -A_3(u^2+v^2+z^2)$, where $A_3 = A_3(a,\delta,\Delta)$.

Proof. From equation (1) and system (2) we have

$$w' = \frac{\partial w}{\partial x}u' + \frac{\partial w}{\partial y}v' + \frac{\partial w}{\partial z}z' = \frac{\partial w}{\partial x}v + \frac{\partial w}{\partial y}z + \frac{\partial w}{\partial z}(-\propto(u,v)z - g(u,v))$$

Which implies that

$$w' = \frac{a\delta}{\Delta}\{[\beta^2(1-\varepsilon)^2uv + \{(1-\varepsilon)[k^2-\beta(1-\varepsilon)+\beta\}vu + z - \propto(u,v)z - g(u,v)\big) + k\beta(1-\varepsilon)^2[v^2+uz] + \big(1-\varepsilon)^2\beta\big[vz + u\big(\propto(u,v)z - g(u,v)\big)\big] + k(1-\varepsilon)\big[z^2 + v\big(\propto(u,v)z - g(u,v)\big)\big]\}.$$

Then we have

$g_u = \frac{g(u,v)-g(0,v)}{u}$ and $g_v = \frac{g(u,v)-g(u,0)}{v}$ after simplification we get

$$w' = \frac{-a\delta}{\Delta}\{u^2+v^2+z^2\}$$

Suppose that $A_3 \le \frac{a\delta}{\Delta}$, then we get this inequality

$$w' = -A_3\{u^2+v^2+z^2\}$$

Here, we arrive to the end of proof of the lemma(1).

Lemma (2): Assume that all assumption of theorem (2) hold, then for any positive constants A_4, A_5 depending on $a,\beta,\varepsilon,k,\Delta,\delta$ for any solution of the system (2) we have

$$w' = \frac{dw(u,v,z)}{dt} \le -A_4(u^2+v^2+z^2) + A_5(|u| + |v| + |z|)|Q(t;u,v,z)|$$

Proof. When $Q \ne 0$

Let $Q(t;u,v,z) = Q(t)$, then we have that

$$w' = \frac{a\delta}{\Delta}\{[\beta^2(1-\varepsilon)^2]uv + \{(1-\varepsilon)[k^2-\beta(1-\varepsilon)+\beta\}vz + z - \propto(u,v)z - g(u,v) + Q(t) + k\beta(1-\varepsilon)^2[v^2+uz] + \big(1-\varepsilon)^2\beta\big[vz + u(\propto(u,v)z - g(u,v) + Q(t))\big] + k(1-\varepsilon)\big[z^2 + v\big(\propto(u,v)z - g(u,v) + Q(t)\big)\}.$$

then we get

$$w' = \frac{dw(u,v,z)}{dt} = -\frac{a\delta}{\Delta}\{u^2+v^2+z^2 - (1-\varepsilon^2\beta)u + k(1-\varepsilon)v + z)Q(t)\} \le -\frac{a\delta}{\Delta}\{u^2+v^2+z^2 - A_4(|u| +$$

v+zQ(t)

$A_4 = maximum((1-\varepsilon)^2\beta, k(1-\varepsilon), 1)$

Therefore $w' = \frac{dw(u,v,z)}{dt} \le -A_3(u^2+v^2+z^2) + A_5(|u|+|v|+|z|)Q(t)$

Where $A_5 = \frac{A_4 a\delta}{\Delta}$, $(|u|+|v|+|z|) \le \sqrt{3}(u^2 + v^2 + z^2)^{\frac{1}{2}}$

Then we get

https://doi.org/10.26682/sjuod.2019.22.1.4

$$\frac{dw(u,v,z)}{dt} \le -A_4(u^2+v^2+z^2)+A_6(u^2+v^2+z^2)^{\frac{1}{2}}Q(t)$$

Where $A_6 = \sqrt{3}A_5$ and $A_3 = A_4$

This complete proof of lemma (2)

Now we give the proof of theorem (1) from the proof of lemma (1) and (2) we conclude that the zero solution of Eq. (1) is asymptotic stable this means that the solutions $(u(t), v(t), z(t))$ of system (2) are satisfy $u^2(t)+v^2(t)+z^2(t) \to 0$ as $t \to \infty$.

For Proof of theorem (2)

From the inequality

$$\frac{dw(u,v,z)}{dt} \le -A_3(u^2+v^2+z^2)+A_6(u^2+v^2+z^2)^{\frac{1}{2}}Q(t)$$

And inequality satisfy the condition $(u^2+v^2+z^2)^{\frac{1}{2}} \le (\frac{2w}{A_1})^{\frac{1}{2}}$, thus it becomes

$$\frac{dw(u,v,z)}{dt} \le -A_7 w + A_8 w^{\frac{1}{2}}|Q(t)|,$$

or

$$w' \le -2A_9 w + A_8 w^{\frac{1}{2}}|Q(t)|,$$

where $A_9 = \frac{1}{2}A_7$, therefore

$$w' + A_9 w \le -A_9 w + A_8 w^{\frac{1}{2}}|Q(t)| \le A_8 w^{\frac{1}{2}}\{|Q(t)| -$$

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Where $A_{10} = \frac{A_9}{A_8}$, this implies that

$$w' + A_9 w \le A_8 w^{\frac{1}{2}} w^*$$

Where $w^* = |Q(t)| - A_{10} w^{\frac{1}{2}} \le w^{\frac{1}{2}}|Q(t)| \le |Q(t)|$

When

$$|Q(t)| \le A_{10} w^{\frac{1}{2}},$$
$$w^* \le \frac{|Q(t)|}{A_{10}}$$

Then we have

$$w' + A_9 w \le A_{11} w^{\frac{1}{2}}|Q(t)|$$

Where

$$A_{11} = \frac{A_8}{A_{10}}$$

This implies

$$w^{-\frac{1}{2}}w' + A_9 w^{\frac{1}{2}} \le A_{11}|Q(t)|$$

By multiplying both sides by $e^{\frac{1}{2}A_9 t}$ we get

$$e^{\frac{1}{2}A_9 t}\{w^{-\frac{1}{2}}w' + A_9 w^{\frac{1}{2}}\} \le e^{\frac{1}{2}A_9 t}A_{11}|Q(t)|$$

$$\frac{d}{dt}\{w^{\frac{1}{2}}e^{\frac{1}{2}A_9 t}\} \le \frac{1}{2}e^{\frac{1}{2}A_9 t}A_{11}|Q(t)|$$

By integrating both sides from t_0 to t we get

$$\{w^{\frac{1}{2}}e^{\frac{1}{2}A_9 s}\}_{t_0}^{t} \le \int_{t_0}^{t}\frac{1}{2}e^{\frac{1}{2}A_9\tau}A_{11}|Q(\tau)|\,d\tau$$

which implies to

$$\{w^{\frac{1}{2}}(t)\}e^{\frac{1}{2}A_9 t} \le w^{\frac{1}{2}}(t_0)e^{\frac{1}{2}A_9 t_0} + \frac{1}{2}A_{11}\int_{t_0}^{t}|Q(\tau)|\,d\tau e^{\frac{1}{2}A_9\tau}d\tau$$

$$w^{\frac{1}{2}}(t) \le e^{-\frac{1}{2}A_9 t}\{w^{\frac{1}{2}}(t_0)e^{\frac{1}{2}A_9 t_0} + \frac{1}{2}A_{11}\int_{t_0}^{t}|Q(\tau)|\,d\tau e^{\frac{1}{2}A_9\tau}d\tau\}$$

and we have this two inequality

$$w \le A_2(u^2+v^2+z^2)$$

and

$$A_1(u^2+v^2+z^2) \le w(u,v,z) \le A_2(u^2+v^2+z^2)$$

by this inequality we get

$$A_1(u^2(t)+v^2(t)+z^2(t)) \le e^{-\frac{1}{2}A_9 t}\{A_2(u^2(t_0)+v^2(t_0)+z^2(t_0))e^{\frac{1}{2}A_9 t_0} + \frac{1}{2}A_{11}\int_{t_0}^{t}|Q(\tau)|\,d\tau e^{\frac{1}{2}A_9\tau}d\tau\}^2$$

$\forall\, t \ge t_0$, this implies that

$$u^2(t)+u'^2(t)+u''^2(t) \le \frac{1}{A_1}\{e^{-\frac{1}{2}A_9 t}\{A_2(u^2(t_0)+v^2(t_0)+z^2(t_0))e^{\frac{1}{2}A_9 t_0} + \frac{1}{2}A_{11}\int_{t_0}^{t}|Q(\tau)|\,d\tau e^{\frac{1}{2}A_9\tau}d\tau\}^2\}$$
$$\le \{e^{-\frac{1}{2}A_9 t}\{k_1 + k_2\int_{t_0}^{t}|Q(\tau)|\,e^{\frac{1}{2}A_9\tau}d\tau\}^2\}$$

where k_1, k_2 are constant depending on $A_1, A_2, (u^2(t_0)+v^2(t_0)+z^2(t_0))$ and A_{11} respectively.

By supposing that $A_9 = \gamma$ we have

$$u^2(t)+v^2(t)+z^2(t) \le \{e^{-\frac{1}{2}\gamma t}\Big\{k_1 + k_2\int_{t_0}^{t}|Q(\tau)|\,e^{\frac{1}{2}\gamma\tau}d\tau\}^2\Big\}.$$

We arrive to the end of proof of theorem (2).

To prove theorem (3) by the assumption of lemma (1) and (2)

$$w' \le -A_4(u^2+v^2+z^2)+A_5(|u|+|v|+|z|)|Q(t)|$$

we get

$$w' \le A_5(|u|+|v|+|z|)^2 a(t)$$

this implies to

$$w' \le A_{11}(u^2+v^2+z^2)a(t)$$

We know that $|u||v| \le \frac{1}{2}(u^2+v^2)$, where $A_{11} = 3A_5$, and from lemma (1) we have

$$w \ge A_1(u^2+v^2+z^2)$$

Then we get

$$w' \le A_{11} w a(t)$$

By integrating both sides from 0 to t we get

$$w(t) - w(0) \le A_{12}\int_0^t w(s)a(s)ds$$

Where

$$A_{12} = \frac{A_{11}}{A_1} = \frac{3A_5}{A_1}$$

Then we have

$$w(t) \leq w(0) + A_{12} \int_0^t w(s)a(s)ds$$

By using Grownwall-Bellman inequality we get

$$w(t) \leq w(0)e^{A_{12} \int_0^t a(s)ds}$$

This complete the proof of theorem (3).

REFERENCE

[1] E.A. Barbashin: Lyapunov functions, Izdat. Nauka, Moscow, 1970.

[2] Ezeilo J.O.C: Further Results of Solutions of a Third Order Differential Equations, Proc. Cambridge Philos. Soc. 59 (1963), 111-116.

[3] Ezeilo J.O.C: A Stability Result for a Certain Third-Order Differential Equations, Ann. Mat. Pura Appl (4) 72 (1966), 1-9.

[4] B.S. Ogundare: Further Results on United Qualitative Properties of Solution of Certain Third Order Non-Linear Differential Equations. IJPAM, Vol.46, No.5,(2008),627-636.

[5] C. Qian: On Global Stability of Third Order Nonlinear Differential Equations. Nonlinear Analysis 42 (2000), 651-661.

[6] Cemil Tunç and Muzaffer Ateş: Stability and Boundedness Results for Solutions of Certain Third Order Nonlinear Vector Differential Equations, Nonlinear Dynamics , ugust 2006, Volume 45, Issue 3–4, pp 273–281.

[7] A.U. AFUWAPE and M.O. OMEIKE: Stability and boundedness of solutions of a kind of third-order delay differential equations*,Volume 29, N. 3, pp. 329–342, 2010, Copyright © 2010 SBMAC, ISSN 0101-8205, Comput. Appl. Math. vol.29 no.3 São Carlos 2010.

[8] Pane M. RemiliD Beldjerd : Stability and ultimate boundedness of solutions of some third order differential equations with delay,Journal of the Association of Arab Universities for Basic and Applied Sciences, Volume 23, June 2017, Pages 90-95.

[9] **A**. T. Ademola and P. O. Arawomo: Stability, Boundedness and periodic solutions to certain second order delay differential equations Obafemi Awolowo University, Nigeria University of Ibadan, Nigeria, Proyecciones Journal of Mathematics, Vol. 36, No 2, pp. 257-282, June 2017. Universidad Cat´olica del Norte Antofagasta – Chile

[10] Lynda D. Oudjedi, Moussadek Remili: Boundedness and stability in third order nonlinear vector differential equations with multiple deviating arguments Journal of the Association of Arab Universities for Basic and Applied Sciences,2017.

أستقرارية ومحدودية الحل المحاذي للمعادلات التفاضلية غير الخطية من الرتبة الثالثة

الخلاصة

في هذا البحث تم أستحدام دالة ليابونوف في دراسة أستقرارية ومحدودية الحل المحاذي للمعادلات التفاضلية غير الخطية من الرتبة الثالثة.

أضافة الى ذلك تم أستخدام طريقة ليابونوف المباشرة من أجل تحديد أستقرارية و محدودية الحل المحاذي للمعادلات التفاضلية وذلك بأيجاد دالة ليابونوف المناسبة لهذا الحل.

الكلمات المفتاحية: غير الخطية , الأستقرار المحاذي , المحدودية , دالة ليابونوف