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A HARVESTED MODIFIED LESLIE-GOWER PREDATOR-PREY MODEL WITH SIS-DISEASE IN PREDATOR AND PREY REFUGE

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ABSTRACT

The aim of this paper is to consider and analyze the dynamic behavior of a modified Leslie-Gower prey-predator model with SIS-disease in predator incorporating prey protection and harvesting factor. The disease is spread from one predator to another through two ways. Physical contact; or external source. Firstly, the details of the assumptions in the proposed model and the significant of the parameters used in are discussed. Then the boundedness of the model is proved, certain conditions for persistence of the model are given and the existence as well as stability analysis of all possible non-negative equilibrium points is studied. Finally, to confirm our analytical finding we discussed numerical simulation of the model.

KEYWORD: Leslie-Gower prey- predator model, prey protection, SIS disease, and Stability analysis.

1. INTRODUCTION

In the mid-1920s, Voltera [19] and Lotka [5] introduce the first mathematical model in ecology, their model describe the dynamics of interaction between predator and prey. Lotka-Voltera Model has been modified during these thirty years [8, 11-12, 15-16].]. Functional response is main aspect of modeling the dynamics of predator-prey interaction because it describes a predator's intendances per capita as function of prey abundance, many authors proposed the lotka-Voletra model with more realistic functional response [1, 4, 10, and 21]. There are various factor which effect positively or negatively on the dynamics between prey and predator like epidemic disease among one species or both, prey refuge which is a type of protection that protect prey species from predation, harvesting species and many other factors[9,17-18]. Leslie and Gower [6] introduced Leslie-Gower prey-predator Model, in their model the carrying capacity of predator depends on to the prey density which is not recognized in Lotka-Volter Model. In case of severe scarcity, the predator species can switch over to other population but its growth will be limited by the fact that it's most favorite food is not available in abundance, therefore, Aziz-Aloui and Daher [2] modified the Leslie-Gower Model which can be written as follows

$$\frac{dX}{dt} = r_1 x\left(1-\frac{X}{K}\right) - \frac{\alpha XY}{1+\alpha Tx}$$
$$\frac{dY}{dt} = r_2 y\left(1-\frac{Y}{X+c}\right) \qquad (1)$$

Where x (t) and y (t) are numbers of prey and predator, respectively. the prey species x (t) grows with intrinsic growth rate r_1 and carrying capacity k and predator grows with intrinsic growth rate r_2 and its carrying capacity is prey density x (t) added by the additional food c which provides protection to the predators and the predators consumes the prey species according Hollying Type functional responses [13-14]. Like the working on modification of lotka-Voltera model by many researcher considered the system (1) including the same factors which taking into account to the lotka - Voltera model[3,7,21-22]and the references therein. Most of this modification included at most two factors, therefore, in this work; we considered and studied the dynamic behavior of a new modified Leslie-Gower prey-predator model with more than two factors. Our proposed model is considered system (1) with SIS-disease in predator species only species and incorporating prey refuge and population harvesting. The disease is transmitted within

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predator species by contact between susceptible predator and infected predator as well as by external source. This paper is structured as follows. In the next section we have discussed the details of the assumptions in the new model and the significant of the parameters used in it, section 3 deals with positive, boundednees and persistence of the model. In section 4 all possible equilibrium points and their existence criteria and discussed stability analysis, in section 5, we give the numerical verification of our analytical finding. Finally, we discussed the conclusion of our work.

2. THE MODEL FORMULATION

The proposed model based on the following assumptions:

1- In the absence of predators and harvesting factor, the prey species follow the Logistic dynamics.
2- An epidemic SIS-disease is transmitted within the predator only by contact directly according to non linear incidence rate, and through an external source, therefore the presence of disease divide the predator species in to two groups: susceptible predator and infected predator.
3- All predators are capable of reproduce susceptible predator and follow logistic dynamics.
4- There is additional source for predator.
5- Not all prey available to be predated, because there is a type of protection of prey from predation (prey refuge).
6- Finally, both prey species and predator species are harvested by external forces.

With above assumption, our model leads to the following set of ordinary differential equation:

$$\frac{dX}{dt} = r_1 X\left(1-\frac{X}{k}\right) - \frac{\alpha_1(1-m)XI}{1+T\alpha_1(1-m)X} - \alpha_2(1-m)XS - h_1 X$$

$$\frac{dS}{dt} = r_2(S+I)\left(1-\frac{S+I}{X+c}\right) - \frac{\lambda_1 IS}{1+I} - \lambda_2 S + aI - h_2 S \qquad (2)$$

$$\frac{dI}{dt} = \frac{\lambda_1 IS}{1+I} + \lambda_2 S - aI - dI - h_3 I$$

Where $X(t)$, S (t) and I (t) represent the densities at time t for prey, susceptible predator and infected predator respectively, with $x(0) \geq 0$, $S(0) \geq 0$, $I(0) \geq 0$ and all the parameters are positive, they have been defined in the Table 1:

Table (1): Biological interpretation of parameters

Parameters	Biological interpretation
r_1	Growth rate of prey population
r_2	Growth rate of susceptible predator
α_1	Predation rate by infected predator
α_2	Predation rate by infected predator
T	Time of handling
K	Carrying capacity of prey species
m	Prey refuge protection rate
λ_1	Infected rate from contact within predators
λ_2	Infected rate from external source
a	Recovering rate of infected predator
c	Additional source for predator species
d	rate of natural death of infected predator
h_1	Prey harvesting rate
h_2	Susceptible predator harvesting rate
h_4	Infected predator harvesting rate

3. BOUNDEDNESS AND PERSISTENCE

the right hand sides of system (2) are continuously differentiable functions in the positive octant, therefore systems (2) has a unique solution. Furthermore in theorem(1) we proved the boundedness of system (2), The persistence of the system (2) is indicated in theorem(2)

Theorem 1. All solutions of the model system (2) that initiate in the state space R_+^3 are uniformly bounded.

Proof: Let $(X(t), S(t), I(t))$ be any solution of the system (2)

the first equation of system (2), it is obtained that:

$$\frac{dX}{dt} \le r_1 X(1 - \frac{X}{k})$$

Then $$\lim_{t\to\infty} Sup X(t) \le K$$

From the system (2), it is obtained that

$$\frac{d(S+I)}{dt} = r_2(S+I)\left(1 - \frac{S+I}{k+c}\right) - h_2 S - dI - h_3 I$$

Thus

$$\frac{d(S+I)}{dt} \le r_2(S+I)\left(1 - \frac{S+I}{k+c}\right)$$

And hence $$\lim_{t\to\infty}(S+I) \le k+c$$

The proof is completed.

Theorem 2. if the following conditions hold, then every solution of model(2) has no zero limit

$$(1-m)(k+c)(\alpha_1+\alpha_2) + h_1 < r_1 \qquad (3)$$

$$\lambda_1 + \lambda_2 + h_2 < r_2 < \frac{a\left(c + \frac{K}{r_1}\left(1 - \left((1-m)(k+c)(\alpha_1+\alpha_2) + h_1\right)\right)\right)}{k+c} \qquad (4)$$

Proof . From system (2), we have

$$\frac{dX}{dt} = r_1 X\left(1 - \frac{X}{k}\right) - \frac{\alpha_1(1-m)X}{1 + \alpha_1 T(1-m)X} I - \alpha_2(1-m)XS - h_1 X$$

and in theorem (1), we have $\lim_{t\to\infty} Sup X(t) \le K$

Thus as time approaches infinity, we obtain that

$$\frac{dx}{dt} \ge r_1 X\left(1 - \frac{X}{k}\right) - \left((1-m)(k+c)(\alpha_1+\alpha_2) + h_1\right)X$$
$$= r_1 X\left(1 - \frac{1}{r_1}\left((1-m)(k+c)(\alpha_1+\alpha_2) + h_1\right) - \frac{X}{k}\right) = r_1 X(\beta - \frac{X}{k})$$

Where $\beta = 1 - \frac{1}{r_1}\left((1-m)(k+c)(\alpha_1+\alpha_2) + h_1\right)$

Condition 3 guarantees that $\lim_{t\to\infty} X(t) \ge \beta k > 0$.

Now since $r_2 S\left(1 - \frac{S+I}{c+X}\right)$ is logistic reproduction of susceptible predator by susceptible predator only and $r_2(S+I)\left(1 - \frac{S+I}{c+X}\right)$ is logistic reproduction of susceptible predator by usceptible predator a swell as infected predator, so $r_2(S+I)\left(1 - \frac{S+I}{c+X}\right) > r_2 S\left(1 - \frac{S+I}{c+X}\right)$.

That is $\frac{ds}{dt} \ge r_2 S\left(1 - \frac{S+I}{c+X}\right) - \lambda_1 S - \lambda_2 S - h_2 S + aI$

And in theorem(1), we have $\lim_{t\to\infty} S \le k+c$

Thus as time approaches infinity, we get

$$\frac{ds}{dt} \ge r_2 S\left(1 - \frac{S}{c+X}\right) + \left(a - \frac{r_2(k+c)}{c+X}\right)I - (\lambda_1 + \lambda_2 + h_2)S$$

If condition(4) holds, then $a - \frac{r_2(k+c)}{c+X} > 0$

That is $\frac{ds}{dt} \geq r_2 S\left(\left(1-\frac{\lambda_1+\lambda_2+h_2}{r_2}\right)-\frac{S}{c+X}\right) \geq r_2 S(\gamma-\frac{S}{c})$, where $\gamma=\left(1-\frac{\lambda_1+\lambda_2+h_2}{r_2}\right)>0$ under condition(4)

Consequently, $\lim_{t\to\infty} S(t) \geq \gamma c>0$ and from the third equation of the model (2), we get $\frac{dI}{dt} \geq \lambda_2 S-(a+d+h_3)I>\lambda_2 \gamma c-(a+d+h_3)I$ and hence $\lim_{t\to\infty} I(t) \geq \frac{\lambda_2\gamma c}{a+d+h_3}>0$

This completes the proof.

4. EQUILIBRIUM POINT AND STABILITY

In this section we will study the existence and stability behavior for each of zero equilibrium point , predator free equilibrium point, prey free equilibrium point and positive equilibrium point of system (2).

4.1 The trivial equilibrium point and axial Equilibrium point.

The Trivial equilibrium point $E_1=(0,0,0)$is always exist while the predator free equilibrium point. $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ exist if the following condition holds

$$h_1<r_1 \qquad (5)$$

The locally asymptotically stable of $E_1=(0,0,0)$ and $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ established in the following theorem.

Theorem 3. Suppose that the following condition holds,

$$r_2< \lambda_2+h_2-\frac{\lambda_2(r_2+a)}{a+d+h_3} \qquad (6)$$

Then

i. The trivial equilibrium point is locally asymptotically stable and the predator free equilibrium point does not exist if $r_1<h_1$

ii. The predator free equilibrium point is locally asymptotically stable if $r_1>h_1$

Proof i. If $r_1<h_1$ then the predator free equilibrium point is negative which is impossible, so it does not exist. And the eigenvalue of the jacobian matrix at $E_1=(0,0,0)$, in the $X-$ direction is negative, while its eigenvalues in $S-$ direction and $I-$ direction are roots for the following equation

$$\gamma^2+(\lambda_2+h_2+a+d+h_3-r_2)\gamma+(\lambda_2+h_2-r_2)(a+d+h_3)-\lambda_2(r_2+a)=0 \qquad (7)$$

Thus both eigenvalues in $S-$ direction and $I-$ direction are negative iff the condition (6) holds. Consequently $E_0=(0,0,0)$ is locally asymptotically stable.

Proof ii. If $r_1>h_1$ then the axial equilibrium point exist, and eigenvalue of jacobian matrix at $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ in the $X-$ direction is negative. The last two eigenvalues of the jacobian matrix at $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ in $S-$ direction and $I-$ direction are roots for the Eq. (7), thus both eigenvalues in $S-$ direction and $I-$ direction are negative iff the condition (6) holds. Consequently $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ is locally asymptotically stable.

The global stability conditions of $E_0=(0,0,0)$ and $E_2=(k\left(1-\frac{h_1}{r_1}\right),0,0)$ are given in theorem (4) and theorem (5), respectively.

Theorem 4. Suppose the following condition hold, then the trivial equilibrium point$E_0=(0,0,0)$ is global stability in R_+^3.

$$max\{r_1,r_2\}<min\{h_1,h_2,d+h_3\} \qquad (8)$$

Proof. From system (2), we have

$\frac{d(X+S+I)}{dt} \leq X(r_1 - h_1)+(r_2 - h_2)S + (r_2 - d - h_3)I \quad \forall(X,S,I) \in Int.R_+^3$
So, from the condition (8), we have
Then $\lim_{t\to\infty}(X + S + I) = 0$
Consequently $E_0 = (0,0,0)$ is globally asymptotically stable.

Theorem 5. If $E_2 = (k\left(1 - \frac{h_1}{r_1}\right), 0,0)$ is locally asymptotically stable and the following condition hold.

$$max\left\{\left(\frac{\alpha_1(1-m)}{1+\alpha_1 TK}\right)^2 \frac{K^2+cK}{2r_1}, \alpha_2{}^2(1-m)^2\frac{K^2+cK}{2r_1}\right\} < r_2 < min\{h_2, d + h_3\} \quad (9)$$

then the equilibrium point $E_2 = (k\left(1 - \frac{h_1}{r_1}\right), 0,0)$ is globally asymptotically stable.

Proof. Consider the following function:

$$V(X,S,I) = X - k\left(1 - \frac{h_1}{r_1}\right) - k\left(1 - \frac{h_1}{r_1}\right) ln\left(\frac{X}{k\left(1-\frac{h_1}{r_1}\right)}\right) + S + I$$

It is easy to see that $V(X,S,I) \in C^1(R_+^3, R)$, in addition $V\left(k\left(1 - \frac{h_1}{r_1}\right), 0,0\right) = 0$, while $V(X,S,I) > 0;\ \forall(X,S,I) \in R_+^3$ and $(X,S,I) \neq \left(k\left(1 - \frac{h_1}{r_1}\right), 0,0\right)$. Further

$$\frac{dV}{dt} \leq -\left(\frac{r_1}{2K}\left(X - k\left(1 - \frac{h_1}{r_1}\right)\right)^2 + \alpha_2(1-m)\left(X - k\left(1 - \frac{h_1}{r_1}\right)\right)S + \frac{r_2}{K+c}S^2\right) - (h_2 - r_2)S$$
$$-\left(\frac{r_1}{2K}\left(X - k\left(1 - \frac{h_1}{r_1}\right)\right)^2 + \frac{\alpha_1(1-m)}{1+\alpha_1 TK}\left(X - k\left(1 - \frac{h_1}{r_1}\right)\right)I + \frac{r_2}{K+c}I^2\right) - (d + h_3 - r_2)I$$

Clearly, under the condition (9), $\frac{dV}{dt}$ is negative definite, and hence the proof is complete.

4.2 The prey free equilibrium point.

The prey free equilibrium point is $E_2 = (0,\bar{S},\bar{I})$, where $\bar{S} = \frac{(a+d+h_3)\bar{I}}{\frac{\lambda_1 \bar{I}}{1+\bar{I}}+\lambda_2}$ and $\bar{I}$ is a root for the following equation

$F(I) = r_2(h(I) + I)\left(1 - \frac{h(I)+I}{c}\right) - \frac{\lambda_1 I h(I)}{1+I} - \lambda_2 h(I) + aI - h_2 h(I) = 0$, where $h(I) = \frac{(a+d+h_3)I}{\frac{\lambda_1 I}{1+I}+\lambda_2}$

In theorem(1) we proved that $\lim_{t\to\infty}(I(t) + S(t)) \leq K + c$, So, if the following condition hold, then by using the intermediate value theorem, $F(I)$has a unique positive root namely $\bar{I} \in (0, K + c)$

$F(0) > 0$ and $F(K + c) < 0$ and $\frac{dF}{dt} < 0$ for all $I \in [0, K + c]$

Now, globally asymptotically stable of the prey free equilibrium point $E_2 = (0,\bar{S},\bar{I})$ is established in the following theorem.

Theorem 6. If the prey free equilibrium point$E_3 = (0,\bar{S},\bar{I})$ exist uniquely, the conditions in theorem (2) and the following condition hold then $E_3 = (0,\bar{S},\bar{I})$ is globally asymptotically stable.

$r_1 < h_1$
(10)
$I_{max} < \frac{c}{r_2}(r_2 + a)$
(11)

Proof. From first equation of the system (2)

$$\frac{dX}{dt} \leq X(r_1 - h_1)$$

Thus from condition (10), we get $\lim_{t\to\infty} x(t) = 0$

Thus , system (2) can be reduced to the following prey free subsystem

$$\frac{dS}{dt} = r_2(S + I)\left(1 - \frac{S + I}{c}\right) - \frac{\lambda_1 IS}{1 + I} - \lambda_2 S + aI - h_2 S = f(S,I)$$

$$\frac{dI}{dt} = \frac{\lambda_1 IS}{1+I} + \lambda_2 S - aI - dI - h_3 I = g(S,I)$$

Consider now the function $H(S,I) = \frac{1}{SI}$, clearly $H : Int.R_+^2 \to R$ which is a continuously differentiable function. Further, since

$$B(S,I) = \frac{\partial}{\partial S}(Hf) + \frac{\partial}{\partial I}(Hg) = \frac{r_2 I - r_2 c - ac}{cS^2} - \frac{r_2}{cI} - \frac{\lambda_1}{(1+I)^2} - \frac{\lambda_1}{I^2}$$

Here, f and g are given in the prey free subsystem. We can say I_{max} represents the upper bound constant for the variable because system(2) is bounded. Therefore, condition (11) guarantees that $B(S,I)$ is not identically zero and it has no change in its sign. So, by Bendixon-Dulac criterion, there is no periodic curve in the $Int.R_+^2$ of the SI −plane.

Now, from conditions in theorem(2), we have $\lim_{t\to\infty} S(t) \geq \gamma c > 0$ and $\lim_{t\to\infty} I(t) \geq \frac{\lambda_2 \gamma c}{a+d+h_3} > 0$, So no of trajectory goes to boundary equilibrium point, and since $(\bar{S},\bar{I})$ is unique positive equilibrium point, so, every trajectory in $Int.R_+^2$ goes to the equilibrium point $(\bar{S},\bar{I})$ Hence the equilibrium point $(\bar{S},\bar{I})$ of the prey free subsystem and then the associated prey free equilibrium point $E_3 = (0,\bar{S},\bar{I})$ of system (2) is globally asymptotically.

4.3 The positive equilibrium point.

The positive equilibrium point $E_{4=}(\check{X},\check{S},\check{I})$ is the solution of the following set of equations.

$$r_1\left(1-\frac{X}{k}\right) - \frac{\alpha_1(1-m)}{1+Th\alpha_1(1-m)X} I - \alpha_2(1-m)S - h_1 = 0$$
$$r_2(S+I)\left(1-\frac{S+I}{X+c}\right) - \frac{\lambda_1 IS}{1+I} - \lambda_2 S + aI - h_2 S = 0$$
$$\frac{\lambda_1 IS}{1+I} + \lambda_2 S - aI - dI - h_3 I = 0$$

Straight forward computation gives that:

$$S = \frac{(a+d+h_3)I}{\frac{\lambda_1 I}{1+I} + \lambda_2} = g_1(I), \quad X = \frac{g_1(I)+I}{r_2(g_1(I)+I) - \frac{\lambda_1 I g_1(I)}{1+I} - \lambda_2 g_1(I) + aI - h_2 g_1(I)} - c = g_2(I)$$

$$G(I) = r_1\left(1-\frac{g_2(I)}{k}\right) - \frac{\alpha_1(1-m)I}{1+Th\alpha_1(1-m)g_2(I)} - \alpha_2(1-m)g_1(I) - h_1 = 0$$

Thus, $\check{S} = g_1(\check{I})$, $\check{X} = g_2(\check{I})$ While $\bar{I} \in (0,k+c)$ represents a positive root of the function $G(I)$

From the intermediate value theorem, $G(I)$ has a unique positive root namely $\bar{I} \in (0,k+c)$, if $G(I)$: $[o,k+c] \to R$ is continuous function with $G(0) > 0 (or\ G(0) < 0)$: $G(k+c) < 0 (or\ G(k+c) > 0)$ and $\frac{dG}{dI} = G'(I) < 0$ for all I $\in [0,k+c]$.

The locally asymptotical stable of the positive equilibrium point $E_{4=}(\check{X},\check{S},\check{I})$ is established in the following theorem

Theorem 7. if the following conditions hold and the positive equilibrium point exist, then $E_{4=}(\check{X},\check{S},\check{I})$ is locally asymptotically stable.

$$\frac{\alpha_1(1-m)\check{X}}{1+T\alpha_1(1-m)\check{X}}+r_1<\frac{2r_1}{k}\check{X}+\frac{\alpha_1(1-m)\check{I}}{\left(1+T_h\alpha_1(1-m)\check{X}\right)^2}+\alpha_2(1-m)\left(\check{S}-\check{X}\right)+h_1 \qquad (12)$$

$$\frac{r_2(\check{S}+\check{I})^2}{(\check{X}+c)^2}+\left|r_2-2r_2\left(\frac{\check{S}+\check{I}}{\check{X}+c}\right)-\frac{\lambda_1\check{S}}{(1+\check{I})^2}+a\right|+r_2<2r_2\left(\frac{\check{S}+\check{I}}{\check{X}+c}\right)+\left(\frac{\lambda_1\check{I}}{1+\check{I}}\right)+\lambda_2+h_2 \qquad (13)$$

$$\frac{\lambda_1\check{I}}{1+\check{I}}+\frac{\lambda_1\check{S}}{(1+\check{I})^2}<a+d+h_3-\lambda_2 \qquad (14)$$

Proof. The variational matrix for system (2)at the point E_4 can be written as
$J(E_4)=(c_{ij})_{3\times3}: i,j=1,2,3$; where

$$c_{11}=r_1-\frac{2r_1\check{X}}{k}-\frac{\alpha_1(1-m)\check{I}}{\left(1+T\alpha_1\left(1-m\check{X}\right)\right)^2}-\alpha_2(1-m)\check{S}-h_1, \qquad c_{12}=-\alpha_2(1-m)\check{X}$$

$$c_{13}=-\frac{\alpha_1(1-m)\check{X}}{1+T\alpha_1(1-m)\check{X}},\quad c_{21}=\frac{r_2(\check{S}+\check{I})^2}{(\check{X}+c)^2},\quad c_{22}=r_2-2r_2\left(\frac{\check{S}+\check{I}}{\check{X}+c}\right)-\left(\frac{\lambda_1 I}{1+\check{I}}\right)-\lambda_2-h_2$$

$$c_{23}=r_2-2r_2\left(\frac{\check{S}+\check{I}}{\check{X}+c}\right)-\frac{\lambda_1\check{S}}{(1+\check{I})^2}+a, c_{31}=0,\ c_{32}=\frac{\lambda_1\check{I}}{1+\check{I}}+\lambda_2\ and\ c_{33}=\frac{\lambda_1\check{S}}{(1+\check{I})^2}-a-d-h_3$$

From theorem of Gerschgorin, the eigenvalues are in the following circles
$|t-c_{11}|=|c_{12}|+|c_{13}|$
$|t-c_{22}|=|c_{21}|+|c_{23}|$
$|t-c_{33}|=|c_{31}|+|c_{32}|$

If all the conditions (12-14) hold then $c_{ii}<0: i=1,2,3$ and
$|c_{11}|>|c_{12}|+|c_{13}|$
$|c_{22}|>|c_{21}|+|c_{23}|$,
$|c_{33}|>|c_{31}|+|c_{32}|$
This means that all the Eigen values are negative
And hence $E_{4=}\left(\check{X},\check{S},\check{I}\right)$ is locally asymptotically stable.

5 NUMERICAL CONFIRM

In this section, we will investigate the dynamics of the system (2) numerically to confirms the analytical finding and discuss the role of the existence of prey refuge and population harvesting on the dynamics of system (2). The numerical solution system (2)at different initial point with the parameters as given in Table 2, is illustrated in Fig. (1).

Table (2): Parameter values for Fig. 1

Parameters	Values
r_1	1.2
r_2	0.8
α_1	0.3
α_2	0.4
T	1
K	100
m	0.5
λ_1	0.5
λ_2	0.2
a	0.1
c	10
d	0.4
h_1	0.5
h_2	0.5
h_3	0.6

Obviously, as shown in Fig. 1, system (2) is persist and approaches asymptotically to the positive equilibrium pointin the (9.3522,2.5992,1.03883) it is easy to verify that the data in Table 2 satisfy the stability conditions (12-14). However for the data set in Table 2 with $m = 0.1$ system (2) approaches asymptotically prey free equilibrium point $(E_3 = 0, 1.6032, 0.5501)$ as shown in Fig. 2.

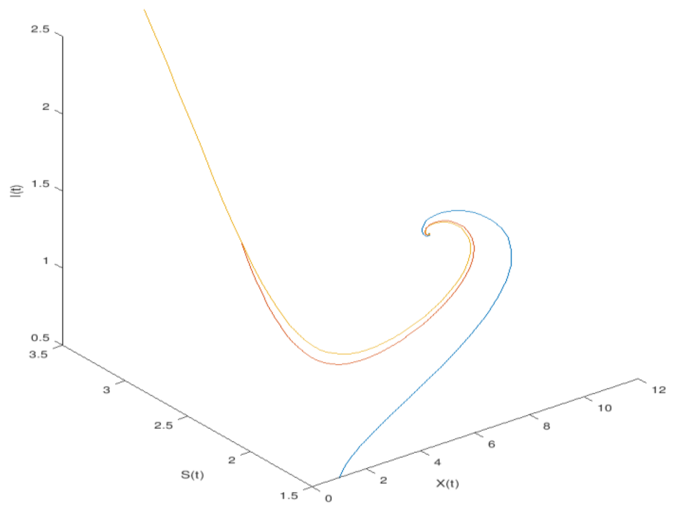

Fig. (1): The phase portrait of the system (2) for the different initial point with parameter values given in Table 2.

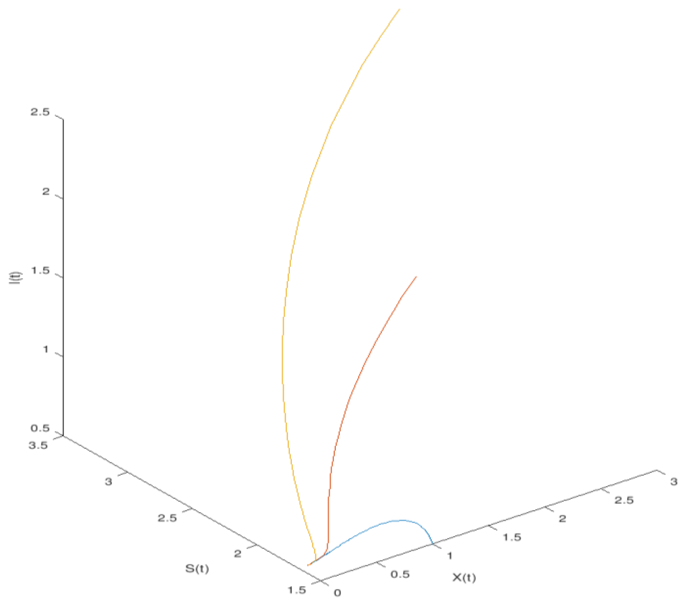

Fig. (2): The phase portrait of the system (2) for the different initial point and $m = 0.1$ with other parameter values given in Table 2.

Note that the data of parameters used in Fig. 2, Satisfy the stability conditions (10)-(11), and hence the above figure confirms the analytical results. Further, it is observed that for the set of data in Table 2 with increasing the predator harvesting rates to $h_2 = 0.9, h_3 = 0.7$, the solution of system (2) approaches asymptotically to the predator free equilibrium point $E_2 = (53.3333,0,0)$ as shown in Fig. 3.

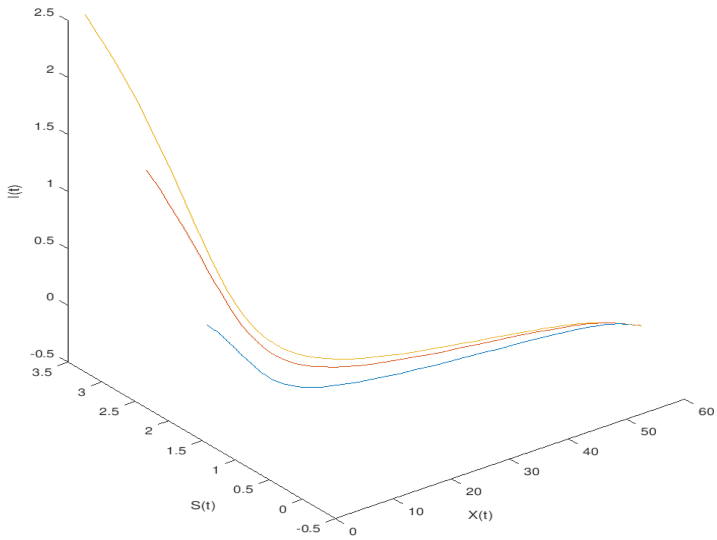

Fig. (3): The phase portrait of the system (2) for the different initial point and $h_2 = 0.9, h_3 = 0.7.$ with other parameter values given in Table 2.

Again, at the data used in Fig. 3 the local stability conditions (5)-(9) hold too. Now, for the data set in Table 2, with $m = 0.1, h_1 = 1.23, \mathrm{h}_2 = 0.9 , h_3 = 0.7,$ the solution of the system (2) approaches asymptotically the trivial equilibrium point $E_1 = (0,0,0)$ as shown in the Fig. 4 below.

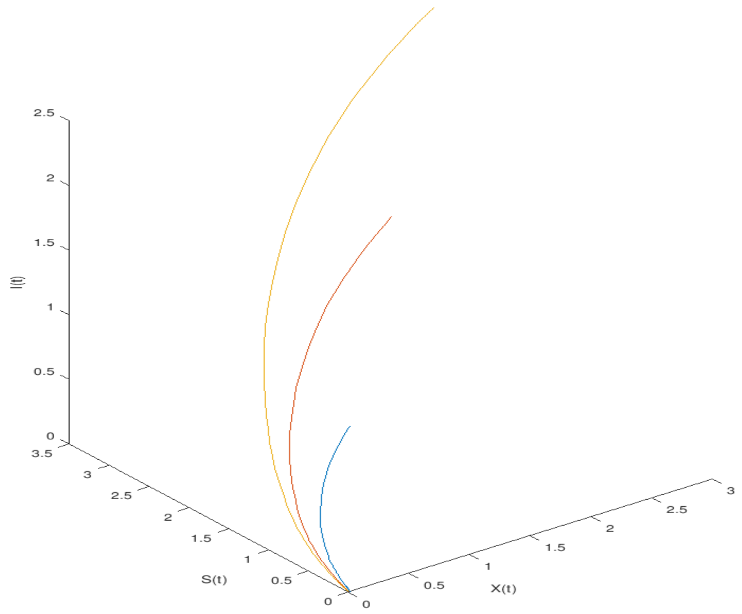

Fig. (4): The phase portrait of the system (2) for the different initial point and $m = 0.1, h_1 = 1.23, h_2 = 0.9$, $h_3 = 0.7$. with other parameter values given in Table 2.

Off course, straight forward computations show that the data set used in the last figure satisfy the stability conditions of the trivial equilibrium point.

6 DISCUSSIONS AND CONCLUSIONS

In this work, we modeled the effect of prey refuge, population harvesting and infection disease on modified the Leslie-Gower prey-predator model and Holling type II functional response. It is assumed that the disease is transmitted through two ways, contact and an external factor, the boundedness of all solution of the system(2)are discussed. The existences of each positive equilibrium points are investigated. Both local as well as global stability analyses for our system are performed. Moreover, in order to confirm our analytical results and discussing the role of parameters used for prey refuge, harvesting rate, infectious rate on the dynamics of system (2), numerical simulations are used for a biologically feasible set of parameters. For the set of data given in Table 2, it is observed that:

1. System (2) persists and the solution initiate at any point in the $Int.R_+^3$ approaches asymptotically to the positive equilibrium point$E_4 = (7.9656,2.5986,1.2390)$ for all values of parameters as given in Table 2

2. if we increase the harvesting rates on predator from $\{h_2 = 0.5, h_3 = 0.6\}$ to $\{h_2 = 0.9, h_3 = 0.7\}$ with the rest of parameter kept as in Table 2 leads to extinction in the predator and prey will be persist, the solution of the system (2) approaches asymptotically to axial equilibrium point $E_2 = (53.3333,0,0)$

3. Decreasing prey refuge rate from 0.5 to 0.1 with the rest of parameters kept fixed as given in Table 2, causes extinction in the prey and then the solution of the system (2) approaches asymptotically to the prey free equilibrium point $E_3 = (0, 1.6032, 0.5501)$, thus the system loses the persistence.

4. Finally, if we increase all population harvesting rates from $h_1 = 0.5, h_2 = 0.5, h_3 = 0.6$ to $h_1 = 0.9, h_2 = 0.9, h_3 = 0.7$ and decreasing prey refuge rate from $m = 0.5$ to $m = 0.1$ with rest of the parameters kept as Table 2, causes extinction in both prey and predator species and system (2) approaches asymptotically to the travail equilibrium point$E_1 = (0,0,0)$.

Consequently, the harvesting rate and prey refuge rate play a vital role in the dynamics of the system (2). In fact, they represent the bifurcation parameters of the system (2).

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