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ON S-TOPOLOGICAL BCK-ALGEBRA

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Abstract. The aim of this paper is to introduce the concept of s-topological BCK-algebras which is a BCK-algebra equipped with a special type of topology that makes the operation defined on it s-topologically continuous.

1. Introduction

Algebras and topology, are two fundamental subjects of pure mathematics. Algebra studies all kinds of operations and topology studies continuity and convergence. The basic principle that describes the relationship between a topology and algebraic operation is to make these operations topologically continuous, perhaps in the first variable, second or jointly continuous which is known as topological algebra. From the beginning of twentieth century many mathematicians have contributed to the development of this subject. After Y. Imai and K. Iseki [7] gave an axiom system of propositional calculus in 1966 and in the same year K. Iseki [8] gave an algebraic formulation for the BCK-propositional calculus system, several mathematicians have been written on the concept of BCK-algebras and found many of the algebraic properties of the BCK-algebras. Besides some of them studied the topological aspects of BCK-algebras. In [1], R. Alo and E. Deeba studied the topological aspects of the BCK-structure in a mode comparable to the study of topological groups, rings and lattices. Lee and Rio in [3] studied the topological structures making the star operation of BCK-algebra continuous and they found nice properties of such BCK-algebras. In 2017, Mehrshad and Golzapoor [12], introduced the concept of topological BE-algebras where BE-algebra is an algebraic structure satisfy some axioms similar to the axioms of BCK-algebras.

In this paper, we study BCK-algebras equipped with certain topologies in which the star operation of the structure satisfied a certain type of continuity, we name this BCK-algebra joined with such topologies by s-topological BCK-algebra. It is proved that every topological BCK-algebra is s-topological

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BCK-algebra. Further, many topological properties of a BCK-algebra were found.

2. PRELEMINARIES

For the development of this paper, we give necessary definitions and properties of a BCK-algebra and investigate the concept of a topological BCK-algebra. For further information, on BCK-algebras we refer to [13].

Definition 2.1. *By a BCK-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in X$,*

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$,
(5) $0 * x = 0$.

In a BCK-algebra $(X, *, 0)$, we define a partial order relation $(\leq)$ by $x \leq y$ if and only if $x * y = 0$.

From the definition of BCK-algebras we can get the following properties.

Proposition 2.1. [4] In a BCK-algebra X, the following statements are true for all $x, y, z \in X$:

(1) $x * 0 = x$,
(2) $x * y \leq x$,
(3) $(x * y) * z = (x * z) * y$,
(4) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$,
(5) $x * (x * (x * y)) = x * y$.

Definition 2.2. [4] *A nonempty subset A of a BCK-algebra $(X, *, 0)$ is called an ideal of X if the following conditions are satisfied:*

(1) $0 \in A$,
(2) *For all $x \in X$ and for all $y \in A$, if $x * y \in A$, then $x \in A$.*

If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called unit of X. A BCK-algebra with unit is called a bounded BCK-algebra [4] .

Definition 2.3. [3] *A BCK-algebra X equipped with a topology τ is called a topological BCK-algebra (for short TBCK-algebra) if $f : X \times X \to X$ defined by $f(x, y) = x * y$ is continuous for all $(x, y) \in X \times X$ where $X \times X$ has the product topology.*
Equivalently, if for each open set O containing $x * y$, there exist open sets U and V containing x and y respectively such that $U * V \subseteq O$.

Definition 2.4. [2] *Let X be a BCK-algebra, and $a \in X$. A left map $L_a : X \to X$ defined by, $L_a(x) = a * x$, for all $x \in X$ and a right map $R_a : X \to X$ by $R_a(x) = x * a$ for all $x \in X$.*
We denote $L(X)$ to be the family of all L_a for all $a \in X$.

Definition 2.5. [2] *A BCK-algebra X is called a positive implicative BCK-algebra, if $(y * x) * (z * x) = (y * z) * x$ for all $x, y, z \in X$.*

For a subset A of a topological space (X, τ), we say that A is regular open if $A = Int(\overline{A})$ and it is semi-open if $A \subseteq \overline{IntA}$. The complement of a semi-open set is called a semi-closed. The closure, interior, semi-closure and semi-interior of A are denoted respectively by $\overline{A}, Int(A), \overline{A}^s$ and $sInt(A)$. For details we refer to [5], [6] and [9].

3. s-topological BCK-algebras

In this section, we introduce the concept of s-topological BCK-algebras and establish some of their properties.

Definition 3.1. *A BCK-algebra X equipped with a topology τ is called an s-topological BCK-algebra(for short SBCK-algebra) if the function $f : X \times X \to X$ defined by $f(x, y) = x * y$ has the property that for each open set O containing $x * y$, there exist an open set U containing x and a semi-open set V containing y such that $U * V \subseteq O$ for all $x, y \in X$.*

Example 3.1. *Let $X = \{0, a, b, c\}$ and $*$ be defined as following : $c * a = c, c * b = c, a * c = a, b * c = b, a * b = 0, b * a = b, x * 0 = x, x * x = 0$ and $0 * x = 0$ for every $x \in X$. $(X, *)$ is given in the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table 1. An SBCK-algebra which is not TBCK-algebra

We can easily check that $(X, 0, *)$ is a BCK-algebra.
Now consider the topology τ on X defined as: $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Then X is not a TBCK-algebra because $b*a = b$, and the open set containing (b, a) is $\{b\} \times X$ is not a subset of the open set $\{b\}$. But it is not difficult to check that X is SBCK-algebra.

Proposition 3.1. *For any subset A of an SBCK-algebra X and any element $x \in X$, the following statements are true:*

1. $\overline{A} * x \subseteq \overline{A * x}$.
2. *If $\overline{A} * x$ is closed, then $\overline{A} * x = \overline{A * x}$.*
3. *In general, the equality in (1) is not true and $\overline{A} * x$ is not closed.*

Proof. (1) Let $y \in \overline{A} * x$ and U be any open set containing y. So $y = a * x$ where $a \in \overline{A}$. Since X is a SBCK-algebra, so there exists an open set V containing a and a semi-open set G containing x such that $V * G \subseteq U$. Also we have $a \in \overline{A}$ implies that $A \cap V \neq \phi$. Suppose that $b \in A \cap V$, so $b * x \in A * x$ and $b * x \in V * x \subseteq V * G \subseteq U$. Hence we obtain that $y \in \overline{A * x}$.

(2) Suppose that $\overline{A} * x$ is closed and let $y \in \overline{A * x}$. If $y \notin \overline{A} * x$, then $y \in (\overline{A} * x)^C$ which is an open set. It is clear that $A * x \subseteq \overline{A} * x$, so we get $A * x \cap (\overline{A} * x)^C = \phi$ which is contradiction and hence the proof is completed.

(3) In Example 3.1, if $A = \{o, a\}$, then $\overline{A} * b = \{0\}$ which is not closed

and $\overline{A * b} = \{0, a\}$, so $\overline{A} * b \neq \overline{A * b}$. □

Proposition 3.2. *For any subset A of an SBCK-algebra X and any element $x \in X$, the following statements are true:*

1. $x * \overline{A}^s \subseteq \overline{x * A}$.
2. *If $x * \overline{A}^s$ is closed, then $x * \overline{A}^s = \overline{x * A}$.*
3. *In general, the equality in (1) is not true and $x * \overline{A}^s$ is not closed.*

Proof. (1) Let $y \in x * \overline{A}^s$ and U be any open set containing y. So $y = x * a$ where $a \in \overline{A}^s$. Since X is a SBCK-algebra, so there exists an open set V containing x and a semi-open set G containing a such that $V * G \subseteq U$. Also we have $a \in \overline{A}^s$ implies that $A \cap G \neq \phi$. Suppose that $b \in A \cap G$, so $x * b \in x * A$ and $x * b \in x * G \subseteq V * G \subseteq U$. Hence we obtain that $y \in \overline{A * x}$.

(2) Suppose that $x * \overline{A}^s$ is closed and let $y \in \overline{x * A}$. If $y \notin x * \overline{A}^s$, then $y \in (x * \overline{A}^s)^c$ which is an open set. It is clear that $x * A \subseteq x * \overline{A}^s$, so we get $(x * A) \cap (x * \overline{A}^s)^c = \phi$ which is contradiction and hence the proof is completed.

(3) In Example 3.1, if $A = \{o, a\}$, then $b * \overline{A}^s = \{b\}$ which is not closed and $\overline{b * A} = \{0, a, b\}$, so $b * \overline{A}^s \neq \overline{b * A}$. □

From Proposition 3.1 and Proposition 3.2, we get the following result:

Corollary 3.1. *For any subset A of an SBCK-algebra X and any element $x \in X$, the following statements are true:*

1. *If $A * x$ is closed, then $\overline{A} * x = A * x$.*
2. *If $x * A$ is closed, then $x * \overline{A}^s = x * A$.*

Proposition 3.3. *For any subsets A and B of an SBCK-algebra X , the following statements are true:*

1. $\overline{A} * \overline{B}^s \subseteq \overline{A * B}$.
2. *If $\overline{A} * \overline{B}^s$ is closed, then $\overline{A} * \overline{B}^s = \overline{A * B}$.*

Proof. (1) Let $x \in \overline{A} * \overline{B}^s$ and U be any open set containing x. So $x = a * b$ where $a \in \overline{A}$ and $b \in \overline{B}^s$. Since X is an SBCK-algebra, so there exists an open set V containing a and a semi-open set G containing b such that $V * G \subseteq U$. Also we have $a \in \overline{A}$ and $b \in \overline{B}^s$, implies that $A \cap V \neq \phi$ and $B \cap G \neq \phi$. Suppose that $a_1 \in A \cap V$ and $b_1 \in B \cap G$, so $a_1 * b_1 \in A * B$ and $a_1 * b_1 \in V * G \subseteq U$. Hence we obtain that $x \in \overline{A * B}$.

(2) Suppose that $\overline{A} * \overline{B}^s$ is closed and let $x \in \overline{A * B}$. If $x \notin \overline{A} * \overline{B}^s$, then $x \in (\overline{A} * \overline{B}^s)^c$ which is an open set in X. It is clear that $A * B \subseteq \overline{A} * \overline{B}^s$, so we get $(A * B) \cap (\overline{A} * \overline{B}^s)^c = \phi$ which is contradiction and hence the proof is completed. □

Proposition 3.4. *In an SBCK-algebra X, if $\{0\}$ is open , then X is discrete.*

Proof. Suppose that $\{0\}$ is open and let x be any point in X. Since $x * x = 0$ for all $x \in X$ and X is SBCK-algebra, so there exists an open set U containing x and a semi-open set G containing x such that $U * G \subseteq \{0\}$. Hence $W = U \cap G$ is a semi-open set containing x. If W contains any other point y, then we obtain that

$x * y = 0$ and $y * x = 0$ which is contradiction. Hence W is a semi-open set contains x only, Therefore, $\{x\}$ is open also. Hence X is discrete. □

Definition 3.2. [11] *A topological space (X, τ) is called semi-T_1 if for each two distinct points $x, y \in X$ there exist two semi-open sets U and V such that U containing x but not y and V containing y but not x.*

Definition 3.3. [11] *A topological space (X, τ) is called semi-T_2 if for each two distinct points$x, y \in X$ there exist two disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.*

Proposition 3.5. *In an SBCK-algebra X, if $\{0\}$ is closed , then X is semi-T_2.*

Proof. Suppose that $\{0\}$ is closed and let x and y be any two distinct points in X, then either $x * y \neq 0$ or $y * x \neq 0$ without loss of generality suppose that $y * x \neq 0$. Hence there exists an open set V containing y and a semi-open set G containing x such that $V * G \subseteq X \setminus \{0\}$. Hence V is open (semi-open) containing x , G is a semi-open set containing y and $V \cap G = \phi$. Hence, we obtain that X is semi-T_2. □

Proposition 3.6. *If the SBCK-algebra $(X, *, \tau)$ is T_0, then it is semi- T_1.*

Proof. Let $x, y \in X$ and $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $x * y \neq 0$. Since X is T_0, there is an open set W containing one of them but not the other. Suppose that W contains $x * y$ and $0 \notin W$.
Since $(X, *, \tau)$ is a semi topological BCK-algebra there exists an open set U containing x and a semi-open set V containing y such that $U * V \subseteq W$. Then U and V are the required semi-open sets containing x and y respectively.
If $0 \in W$ and $x * y \notin W$. Then we have, $x * x = 0 \in W$, so there exists an open set U containing x and a semi-open set V containing x such that $U * V \in W$, and $y * y = 0 \subseteq W$, there exists an open set U_1 containing y and a semi-open set V_1 containing y such that $U_1 * V_1 \subseteq W$.
Therefore, $G = U \cap V$ and $H = U_1 \cap V_1$ are two semi-open sets containing x and y respectively. It is clear that $y \notin G$ and $x \notin H$.
Hence $(X, *, \tau)$ is a semi-T_1 space. □

Proposition 3.7. *If Y is an open BCK-subalgebra of an SBCK-algebra X, then Y is also an SBCK-algebra.*

Proof. Let $x, y \in Y$ and let U be any open set in the subspace Y containing $x * y$, then there exists an open set V in X containing $x * y$ such that $U = Y \cap V$. Since X is an SBCK-algebra, so there exists an open set W in X containing x and a semi-open set G in X containing y such that $W * G \subseteq V$. But then the intersections $O = W \cap Y$ is an open set in Y containing x and $H = G \cap Y$ is a semi-open set in Y containing y, we have $(W \cap Y) * G \cap Y = (W * G) \cap Y \subseteq V \cap Y = U$. Hence we get the proof. □

Proposition 3.8. *If A is an ideal in an SBCK-algebra X and $0 \in Int(A)$, then A is open.*

Proof. Let $x \in A$. Since $0 \in Int(A)$ and $x * x = 0$, so there is an open set U such that $0 \in U \subseteq A$. Since

X is an SBCK-algebra, so there exists an open set V containing x such that $V * x \subseteq U$. If there is a point $y \in V \cap (X \setminus A)$, so we obtain that $y * x \in A$. Since $x \in A$ and A is an ideal, so $y \in A$ which is contradiction. Hence $x \in V \subseteq A$ implies that A is open. □

Proposition 3.9. *If A is an open ideal in an SBCK-algebra X, then A is semi-closed and hence it is regular open.*

Proof. Let $x \notin A$. Then there exists an open set V containing x and a semi-open set U containing x such that $V * U \subseteq A$. since $x * x = 0$. Hence, if $W = V \cap U$, then we have W is a semi-open set containing x and $W * W \subseteq A$. If some $y \in (W \cap A)$ and since A is an ideal, then we obtain that $W \subseteq A$. This is contradiction. Hence $W \subseteq X \setminus A$ and therefore, A is semi-closed. Since A is open , so we obtain that $A \subseteq Int(\overline{A}) \subseteq A$ which implies that $A = Int(\overline{A})$. Hence A is regular open. □

Definition 3.4. *Let $(X, *, 0)$ be a SBCK-algebra and $F \subseteq X$. Then we say that F is a filter when it satisfies the conditions:*

(1) $0 \in F$,
(2) *If $0 \neq x \in F$ and $x * y \in F$, then $y \in F$.*

Example 3.2. *Let $X = \{0, a, b, c\}$ and $*$ be defined as in the following table:*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	a	a	0

Table 2. A BCK-algebra which contains non-trivial filters

It can be easily seen that both $\{0, b\}$ and $\{0, c\}$ are filters but $\{0, a\}$ is not.

Proposition 3.10. *Let $(X, *, \tau)$ be SBCK-algebra and F be a filter on X. If 0 is an interior point of F, then F is semi-open.*

Proof. Suppose that 0 is an interior point of F. Then there exists an open set U containing 0 such that $U \subseteq F$. Let $x \in F$ be an arbitrary element. Since $x * x = 0$, there exists an open set V containing x and a semi-open set W containing x such that $V * W \subseteq U \subseteq F$. Now, for each y in a semi-open set W, we have $x * y \in F$. Since F is a filter and $x \in F$, we have $y \in F$. Hence $x \in W \subseteq F$ and so F is semi-open. □

Proposition 3.11. *Let $(X, *, \tau)$ be a SBCK-algebra and F be a filter of X. If F is open, then it is closed.*

Proof. Let F be a filter of X which is open in (X, τ). We show that $X \setminus F$ is open. Let $x \in X \setminus F$. Since F is open, 0 is an interior point of F.

Since $x * x = 0$, there exists an open set V containing x and a semi-open set W containing x such that $V * W \subseteq F$. We claim that $V \subseteq X \setminus F$. If $V \not\subseteq X \setminus F$, then there exists an element $y \in V \cap F$. For each $z \in W$, we have $y * z \in V * W \subseteq F$, since $y \in F$ and F is a filter, $z \in F$. Hence $W \subseteq F$ and so $x \in F$ which is contradiction. Therefore, $x \in V \subseteq X \setminus F$ which implies that $X \setminus F$ is open and hence, F is closed. □

Corollary 3.2. *If F is a non-trivial open filter in the SBCK-algebra X, then X is topologically disconnected.*

Definition 3.5. *Let X be a BCK-algebra, U be a non-empty subset of X and $a \in X$. The subsets U_a and ${}_aU$ are defined as follows:*
$U_a = \{x \in X : x * a \in U\}$ and ${}_aU = \{x \in X : a * x \in U\}$. Also if $K \subseteq X$ we put ${}_KU = \cup_{a \in K}\, {}_aU$ and $U_K = \cup_{a \in K} U_a$.

Proposition 3.12. [4] Let X be a BCK-algebra and A, B, W, K be a non-empty subsets of X then:
(1) If $A \subseteq B$, then ${}_AW \subseteq {}_BW$.
(2) If $W \subseteq K$, then ${}_AW \subseteq {}_AK$.
(3) If $F \subseteq X$, then $(F_a)^c = (F^c)_a$ and $({}_aF)^c =_a (F^c)$ for each $a \in X$.

Proposition 3.13. *Let X be an SBCK-algebra, U and F be two non-empty subsets of X, the following statements are true:*
(1) If U is open, then U_a is open and ${}_aU$ is semi-open.
(2) If F is closed, then F_a is closed and ${}_aF$ is semi-closed.

Proof. (1) Let U be an open set, $a \in X$ and let $x \in U_a$. Then $x * a \in U$. Since X is SBCK-algebra, then there exist an open set G containing x and a semi-open set A containing a such that $G * A \subseteq U$, $x * a \in G_a \subseteq U$, thus $G * a \subseteq U$. Then $x \in G \subseteq U_a$. So U_a is open .
To prove that ${}_aU$ is semi-open, let $x \in {}_aU$ implies that $a * x \in U$. Since X is SBCK-algebra, then there exist an open set A containing a and a semi-open set H containing x such that $A * H \subseteq U$, so $a * x \in {}_aH \subseteq U$, thus $a * H \subseteq U$. Hence, $x \in H \subseteq {}_aU$. Therefore, ${}_aU$ is semi-open.

(2) Let F be closed, then F^c is open. Hence, by (1), $(F^c)_a$ is open and ${}_a(F^c)$ is semi-open. By Proposition 3.12, $(F_a)^c = (F^c)_a$ and $({}_aF)^c =_a (F^c)$. Hence, $(F_a)^c$ is open and $({}_aF)^c$ is semi-open. Consequently, F_a is closed and ${}_aF$ is semi-closed. □

Definition 3.6. [4] *Let X be a BCK-algebra. The binary operation $\odot$ will be defined on $L(X)$ as $(L_a \odot L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.*

Theorem 3.1. *Let X be a positive implicative BCK-algebra, then $(L(X), \odot, L_0)$ is a BCK-algebra.*

Proof. Let L_a and L_b be any two elements of $L(X)$. Then, by definition $(L_a \odot L_b)(x) = L_a(x) * L_b(x) = (a * x) * (b * x)$. Since X is positive implication BCK-algebra, so $(a * x) * (b * x) = (a * b) * x$. Hence, $(L_a \odot L_b)(x) = L_{a*b}(x)$ which implies that $L_a \odot L_b = L_{a*b}$ for all $a, b \in X$. Now

(1) $((L_x \odot L_y) \odot (L_x \odot L_z)) \odot (L_z \odot L_y) = (L_{x*y} \odot L_{x*z}) \odot L_{z*y} = L_{((x*y)*(x*z))*(z*y)} = L_0$
(2) $(L_x \odot (L_x \odot L_y)) \odot L_y = (L_x \odot (L_{x*y}) \odot L_y = L_{(x*(x*y)} \odot L_y = L_{(x*(x*y))*y} = L_0$
(3) $L_x \odot L_x = L_{x*x} = L_0$

(4) $L_x \odot L_y = L_0$ and $L_y \odot L_x = L_0$, then $L_{x*y} = L_0$ and $L_{y*x} = L_0$ which implies that $x * y = 0$ and $y * x = 0$ $\Rightarrow x = y$ and hence, $L_x = L_y$.

(5) $L_0 \odot L_x = L_{0*x} = L_0$.

Hence, $L(X)$ is a BCK-algebra. □

Definition 3.7. [4] *Let X be a BCK-algebra, we define a map $\Phi : X \to L(X)$ by $\Phi(x) = L_x$ for all $x \in X$ and if A is any subset of X, then $L_A = \{L_a : a \in A\}$.*

Remark 3.1. *If X is a positive implicative BCK-algebra, then the following statements can be easily proved.*

(1) *If $A \subseteq B$, then $\Phi(A) \subseteq \Phi(A)$.*

(2) *If A and B are any two subsets of X, then $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ and $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.*

Proposition 3.14. *Let X be a positive implicative BCK-algebra, then the map $\Phi : X \to L(X)$ is a BCK-isomorphism.*

Proof. It is clear that Φ is a bijection. We have $\Phi(x * y) = L_{x*y}$ and $L_{x*y}(z) = (x * y) * z$. Since X is positive implicative, we have $(x * y) * z = (x * z) * (y * z)$. Therefore, $L_{x*y}(z) = L_x(z) \odot L_y(z) = (L_x \odot L_y)(z)$. Hence, $\Phi(x * y) = \Phi(x) \odot \Phi(y)$ for all $x, x \in X$, so Φ is a BCK-isomorphism. □

Proposition 3.15. *Let X be a positive implicative BCK-algebra and τ be a topology on X, then the following statements are true:*

(1) *The family $\sigma = \{\Phi(G) \subseteq L(X) : G \in \tau\}$ is a topology on $L(X)$.*

(2) *For any subset A of X, $L_{\overline{A}} = \overline{L_A}$.*

(3) *If A is any semi-open set in (X, τ), then $\Phi(A)$ is a semi-open set in $(L(X), \sigma)$.*

Proof. (1) The proof of σ is a topology and hence it is obvious.

(2) For any subset A of X, we have $A \subseteq \overline{A}$. Hence, $L_A \subseteq L_{\overline{A}}$ and $\overline{A}$ is closed in X, so by definition of $\sigma L_{\overline{A}}$ is closed in $L(X)$. Therefore, we obtain $\overline{L_A} \subseteq \overline{L_{\overline{A}}} = L_{\overline{A}}$. To prove $L_{\overline{A}} \subseteq \overline{L_A}$, let $L_x \in L_{\overline{A}}$, then $x \in \overline{A}$ and let L_G be any open set containing L_x. Hence G is an open set containing x, hence $A \cap G \neq \phi$. Therefore, $L_A \cap L_G \neq \phi$. Implies that $L_x \in \overline{L_A}$, so $L_{\overline{A}} \subseteq \overline{L_A}$ and hence $L_{\overline{A}} = \overline{L_A}$.

(3) Let A be any semi-open set in X, so there exists an open set O in X such that $O \subseteq A \subseteq \overline{O}$. Hence $L_O \subseteq L_A \subseteq L_{\overline{O}}$ and by (2), $L_O \subseteq L_A \subseteq \overline{L_O}$. Hence, L_A is semi-open in $L(X)$. □

Proposition 3.16. *Let X be a positive implicative SBCK-algebra. Then $(L(X), \odot, \sigma)$ is an SBCK-algebra.*

Proof. Let L_W be an open set containing $L_x \odot L_y = L_{x*y}$. Hence W is an open set containing $x * y$ in X and since X is an SBCK-algebra, so there exist an open set U and a semi-open set V containing x and y respectively and $U * V \subseteq W$. Therefore, $L_{U*V} \subseteq L_W$. Since X is positive implicative, so $L_{U*V} = L_U \odot L_V \subseteq L_W$. By Proposition 3.15, L_V is semi-open in $L(X)$ containing L_y, hence the proof is completed. □

Recalling that a function $f : X \to Y$ is semi-continuous [9] if the inverse image of each open set in Y is a semi-open set in X, and it is semi-open if the image of each open set is semi-open.

Proposition 3.17. Let X be an SBCK-algebra, then every left map on X is semi-continuous.

Proof. Let $a \in X$, define a left map $L_a : X \to X$ by $L_a(x) = a * x$, for all $x \in X$. Let W be any open set containing $L_a(x) = a * x$. Since X is an SBCK-algebra, so there exists an open set U containing a and a semi-open set V containing x such that $U * V \subseteq W$. clearly, $a * V \subseteq U * V \subseteq W$. Hence, $L_a(V) \subseteq W$. This implies that L_a is semi continuous. □

Definition 3.8. [4] *A BCK-algebra X is called s-transitive (resp.,s-open) if for each $a \in X \setminus \{0\}$, the left map L_a is semi-continuous (resp., semi-open) and it is transitive open if the right map R_a is both continuous and open. .*

Remark 3.2. From Proposition 3.17, if X is an SBCK-algebra such that for each $a \in X \setminus \{0\}$, the left map L_a is semi-open, then X is s-transitive and s-open.

Proposition 3.18. Let X be an SBCK-algebra such that for each $a \in X \setminus \{0\}$, the left map L_a is semi-open. If U is an open subset of X, then the following statements are true:

(1) The set $a * U$ is semi-open.
(2) $L_a^{-1}(U) = \{x \in X : a * x \in U\}$ is semi-open.
(3) The set $A * U$ is semi-open for each $A \subseteq X$.

Proof. Since L_a is semi-open and U is open, so $L_a(U) = a * U$ is semi-open. By Proposition 3.17, L_a is semi-continuous. Hence $L_a^{-1}(U) = \{x \in X : a * x \in U\}$ is semi-open. Lastly, we have $A * U = \cup_{a \in A}(a * U)$ is the union of semi-open sets, so $A * U$ is semi-open. □

Proposition 3.19. *Let X be an SBCK-algebra, then every right map on X is continuous.*

Proof. Let $a \in X$, define a right map $R_a : X \to X$ by $R_a(x) = x * a$, for all $x \in X$. Let W be any open set containing $R_a(x) = x * a$. Since X is an SBCK-algebra, so there exists an open set U containing x and a semi-open set V containing a such that $U * V \subseteq W$. clearly, $U * a \subseteq U * V \subseteq W$. Hence, $R_a(U) \subseteq W$. This implies that R_a is continuous. □

Proposition 3.20. *Let U be an open subset of a transitive open SBCK-algebra X and let $a \in X$. Then the following statements are true:*

(1) *The set $U * a$ is open.*
(2) *$R_a^{-1}(U) = \{x \in X : x * a \in U\}$ is open.*
(3) *The set $U * A$ is open for each $A \subseteq X$.*

Proof. Since R_a is open and U is open, so $L_a(U) = U * a$ is open. By Proposition 3.19, R_a is continuous. Hence $R_a^{-1}(U) = \{x \in X : a * x \in U\}$ is open. Lastly, we have $U * A = \cup_{a \in A}(U * a)$ is the union of open sets, so $U * A$ is open. □

Definition 3.9. *A BCK-algebra X is called an edge BCK-algebra, if for each $x \in X$ the set $x * X = \{0, x\}$.*

It is clear that Example 3.1 is an edge BCK-algebra.

Proposition 3.21. *Let X be any s-transitive s-open edge BCK-algebra* *and τ be any topology on X, then there exists a topology σ on X which is SBCK-algebra.*

Proof. Let $x \in X \setminus \{0\}$, then L_x is s-open map. Since $X \in \tau$, so we have by Proposition 3.1, $L_x(X)$ is a semi-open set. Hence $L_x(X) = x * X = \{0, x\}$ because X is an edge BCK-algebra. Therefore, we obtain that $\{0, x\}$ is semi-open in X for all $x \in X$. Since, $\{0, x\}$ is semi-open in X for all $x \in X$, so $int(\{0, x\}) \neq \phi$ for all $x \in X$. Therefore, we get the following cases:

Either $int(\{0, x\}) = \{0\}$ or $int(\{0, x\}) = \{0, x\}$ or $int(\{0, x\}) = \{x\}$ for all $x \in X$. In the first two cases we obtain that $\{0\}$ is an open set, so σ is the discrete topology. The last case gives us $\{x\}$ is open for all $x \in X \setminus \{0\}$. We claim that X equipped with the topology σ is an SBCK-algebra. For this, let U be any open set containing $x * y$. If $x \neq 0$ and $y \neq 0$, then $\{x\}$ and $\{y\}$ are open sets containing x and y respectively, so $\{x\} * \{y\} \subseteq U$. If $x = 0$, then $x * y = 0$ and and hence if U is any open set containing 0, we obtain that $U * \{y\} \subset U$. If $y = 0$, then $x * 0 = x$ and $U = \{x\}$ is open and if y is any element of X such that $x > y$, then $\{0, y\}$ is a semi-open set containing 0 and $\{x\} * \{0, y\} \subseteq \{x\}$. Hence X is an SBCK-algebra. □

4. CONCLUSION

In this paper, we defined a special type of topology on the BCK-algebra. Also we introduced the concept of S-topological BCK-algebra as a generalization of the concept of topological BCK-algebra and investigate some of its properties.

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